Problem 23) a) $\vec{H}(t) = J_{s0} \sin(2\pi f t)\,\hat{z}$ ← Same as surface current density, but at 90°. (Use the right-hand rule.)

$\vec{B}(t) = \mu_0 \vec{H}(t) = \mu_0 J_{s0} \sin(2\pi f t)\,\hat{z}$

Outside the solenoid the $\vec{H}$ and $\vec{B}$ fields are approximately zero, because the frequency f is assumed to be small.

b) Total magnetic flux $\Phi = \pi R_1^2 B(t) = \mu_0 \pi R_1^2 J_{s0} \sin(2\pi f t)$

Faraday's law: $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \Rightarrow \oint_{loop} \vec{E} \cdot d\vec{\ell} = -\frac{\partial}{\partial t}\Phi$

The Voltage $V(t)$ is defined in the figure such that $\oint \vec{E} \cdot d\vec{\ell}$ is taken in the clockwise direction (when seen from above). This introduces another minus sign into the equation. Consequently,

$$V(t) = -\oint \vec{E} \cdot d\vec{\ell} = \frac{\partial}{\partial t}\Phi = 2\pi^2 \mu_0 f R_1^2 J_{s0} \cos(2\pi f t)$$

c) In Assignment #3, Problem 6, we saw that the vector potential $\vec{A}(\vec{r})$ of an infinitely long solenoid at a radius $r > R_1$ is given by:

$$\vec{A}(\vec{r}) = \tfrac{1}{2}(\mu_0 J_s R_1^2 / r)\,\hat{\phi}$$

In the present quasi-static problem, $J_s = J_{s0} \sin(2\pi f t)$ and $r = R_2$.

Therefore, $\vec{A}(\vec{r}) = \frac{1}{2}\mu_0 (R_1^2/R_2) J_{s0} \sin(2\pi f t)\,\hat{\phi}$.

Since $\vec{\nabla} \cdot \vec{J} = 0$ for the surface current running around the solenoid there are no free charges in this problem (i.e., ρ or $\sigma_s = 0$). The scalar potential ψ is therefore zero. Consequently,

$$\vec{E} = -\vec{\nabla}\psi - \frac{\partial \vec{A}}{\partial t} = -\frac{\partial \vec{A}}{\partial t} = -\pi \mu_0 f (R_1^2/R_2) J_{s0} \cos(2\pi f t)\,\hat{\phi} \leftarrow \text{at } r = R_2$$

$$V(t) = -\oint \vec{E} \cdot d\vec{\ell} = -2\pi R_2 E = 2\pi^2 \mu_0 f R_1^2 J_{s0} \cos(2\pi f t). \checkmark$$