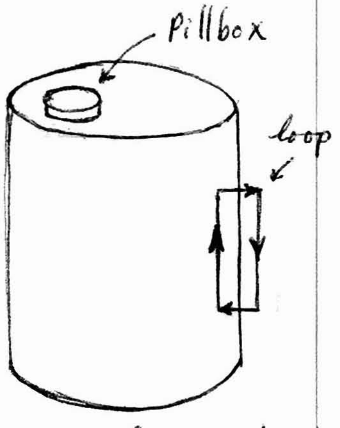
Problem 22) a) A thin pillbox can be used at the top and bottom surfaces of the cylinder to determine $\vec{\nabla}\cdot\vec{M}$. Let the pillbox have surface area S and thickness τ. Then $\int_{\text{Pillbox}} \vec{M}\cdot d\vec{s} = -M_0 S$.



Dividing by the volume $S\tau$, we find $\vec{\nabla}\cdot\vec{M} = -M_0/\tau$.
Thus, at the top of the cylinder, $\rho_m = -\vec{\nabla}\cdot\vec{M} = M_0/\tau$. Since τ can be arbitrarily small, we use $\sigma_{sm} = \rho_m \tau$ instead. Therefore, at the top, $\sigma_{sm} = M_0$.
Similarly, at the bottom, $\sigma_{sm} = -M_0$.

To find $\vec{\nabla}\times\vec{M}$, we use a narrow rectangular loop at the cylinder's sidewall. Again the width w of this loop must approach zero. The integral around the loop is $\oint \vec{M}\cdot d\vec{\ell} = M_0 \ell$, where ℓ is the length of the rectangle. Dividing by the loop area, ℓw, yields $\vec{\nabla}\times\vec{M} = (M_0/w)\hat{\phi}$. Since w can be arbitrarily small, we'll use $\vec{J}_{sm} = \vec{J}_m w$ instead. Consequently, $\vec{J}_{sm} = M_0\hat{\phi}$ on the cylinder's sidewall.

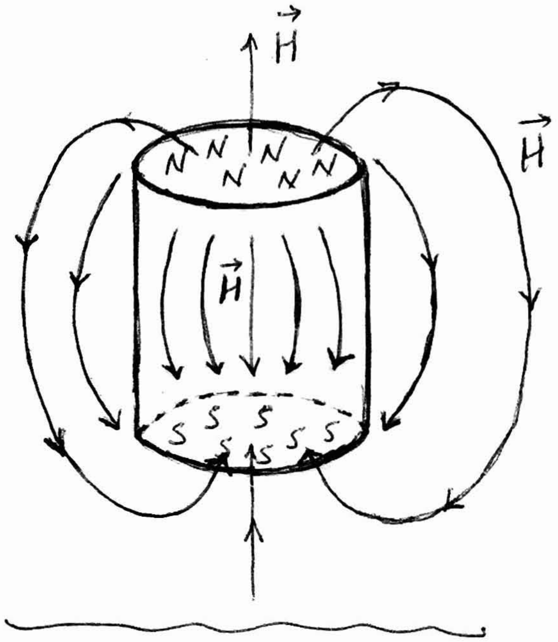
b) $\vec{B} = \mu_0(\vec{H}+\vec{M})$ and $\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow \vec{\nabla}\cdot(\vec{H}+\vec{M}) = 0 \Rightarrow \vec{\nabla}\cdot\vec{H} = -\vec{\nabla}\cdot\vec{M} \Rightarrow$
$\vec{\nabla}\cdot\vec{H} = \rho_m$. This is similar to the first Maxwell equation, $\vec{\nabla}\cdot\vec{D} = \rho_e$, where ρ_e is the density of free electric charges. Here $\vec{D} = \epsilon_0\vec{E} + \vec{P}$ is the displacement field. In the absence of $\vec{P}$, we'll have $\vec{D} = \epsilon_0\vec{E}$.
In electrostatics, $\vec{\nabla}\times\vec{E} = 0$, which, in the absence of $\vec{P}$, yields: $\vec{\nabla}\times\vec{D} = 0$.
Thus, in the absence of $\vec{P}$, the equations that determine $\vec{D}$ are $\vec{\nabla}\cdot\vec{D} = \rho_e$ and $\vec{\nabla}\times\vec{D} = 0$ in the electrostatic regime.

In magnetostatics, when there is a distribution of magnetization, $\vec{M}(\vec{r})$, but no free currents (i.e., $\vec{J}_{\text{free}} = 0$), we'll have $\vec{\nabla}\cdot\vec{H} = \rho_m$ and $\vec{\nabla}\times\vec{H} = 0$.

The situation, therefore, is similar to that in electrostatics for $\vec{D}$. The magnetic charge density ρ_m (or σ_{ms}) gives rise to the $\vec{H}$-field in exactly the same way that the electric charge density ρ_e (or σ_{es}) would give rise to the D-field. The lines of $\vec{H}$ for the cylindrical permanent magnet are shown below.

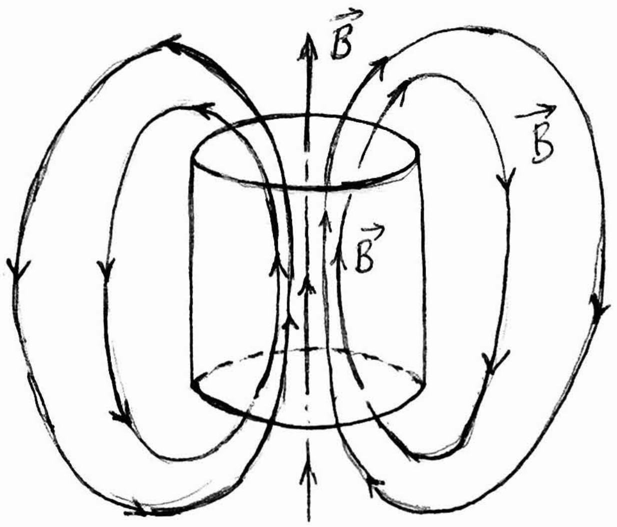
✓Charge density at the top (north pole) = M_o

✓Charge density at the bottom (South Pole) = $-M_o$



c) The magnetization $\vec{M}(\vec{r})$ is produced by the magnetic current distribution $\vec{J}_m(\vec{r}) = \vec{\nabla}\times\vec{M}(\vec{r})$. This current produces a B-field in accordance with the Biot-Savart law, namely, $\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}_m(\vec{r}\,')\times(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\,dv'$.

In the case of the cylindrical permanent magnet, the current $\vec{J}_m$ is a surface current, $\vec{J}_{ms}(\vec{r}) = M_o\hat{\Phi}$ on the cylinder's sidewall. The situation therefore, is the same as a solenoid of length L and radius R, with a uniform surface current $\vec{J}_{ms} = M_o\hat{\Phi}$. The B-field profile of this permanent magnet is shown in the figure →



Inside the magnet $\vec{B}=\mu_0(\vec{H}+\vec{M})$, with $\vec{M}=M_0\hat{z}$. Outside the magnet, $\vec{B}=\mu_0\vec{H}$.

In general, it is possible to prove that, for a static magnetization distribution $\vec{M}(\vec{r})$, the magnetic charge density $\rho_m=-\vec{\nabla}\cdot\vec{M}$ produces the scalar magnetic potential $\psi(\vec{r})=\frac{1}{4\pi}\int\frac{\rho_m(\vec{r}')}{|\vec{r}-\vec{r}'|}dv'$, from which one obtains the magnetic H-field as follows: $\vec{H}(\vec{r})=-\vec{\nabla}\psi=\frac{1}{4\pi}\int(\vec{\nabla}\cdot\vec{M})\vec{\nabla}(\frac{1}{|\vec{r}-\vec{r}'|})dv'$. Similarly, the magnetic current density $\vec{J}_m=\vec{\nabla}\times\vec{M}$ produces the B-field via the Biot-Savart law as follows: $\vec{B}(\vec{r})=\frac{-\mu_0}{4\pi}\int(\vec{\nabla}\times\vec{M})\times\vec{\nabla}(\frac{1}{|\vec{r}-\vec{r}'|})dv'$. (The minus sign is due to the fact that $\vec{\nabla}(\frac{1}{|\vec{r}-\vec{r}'|})=-\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}$.) One can show that, in general,

$$\text{(I)}\qquad \int_{\text{all space}}\left[(\vec{\nabla}\cdot\vec{M})\vec{\nabla}\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)+(\vec{\nabla}\times\vec{M})\times\vec{\nabla}\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)\right]dv'=-4\pi\vec{M}(\vec{r}),$$

which proves that the $\vec{H}$ and $\vec{B}$ thus calculated for a given $\vec{M}(\vec{r})$ satisfy the relation $\vec{B}(\vec{r})=\mu_0[\vec{H}(\vec{r})+\vec{M}(\vec{r})]$. One way to prove Eq. (I) alone is by Fourier transforming the equation and demonstrating its validity in the $\vec{k}$-space.