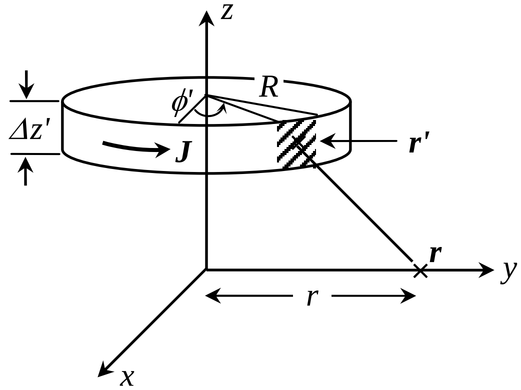
Problem 18)



$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_o}{4\pi}\int_{-\infty}^{\infty}\frac{\boldsymbol{J}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}\boldsymbol{r}',$$

$$|\boldsymbol{r}-\boldsymbol{r}'| = \sqrt{(x-x')^2+(y-y')^2+(z-z')^2}$$

$$= \sqrt{(R\cos\phi')^2+(r-R\sin\phi')^2+z'^2}$$

$$= \sqrt{R^2+r^2-2rR\sin\phi'+z'^2}\,.$$

From symmetry, it is clear that the vector potential $\boldsymbol{A}$ is everywhere in the $\hat{\boldsymbol{\phi}}$ direction, and that the contribution of the volume element shown in the figure is $J_{so}\sin\phi' R\Delta\phi'\Delta z'$. Therefore,

$$A_\phi(r) = \frac{\mu_o}{4\pi}\int_{\phi'=0}^{2\pi}\int_{z'=-\infty}^{\infty}\frac{J_{so}R\sin\phi'}{\sqrt{R^2+r^2-2rR\sin\phi'+z'^2}}\mathrm{d}\phi'\mathrm{d}z'$$

$$= \frac{\mu_o J_{so}R}{2\pi}\int_{\phi'=0}^{2\pi}\sin\phi'\mathrm{d}\phi'\int_{z'=0}^{\infty}\frac{\mathrm{d}z'}{\sqrt{R^2+r^2-2rR\sin\phi'+z'^2}}\,.$$

Depending on the value of ϕ', the expression $(R^2+r^2-2rR\sin\phi')$ is somewhere between $(R-r)^2$ and $(R+r)^2$, thus ensuring that it is always non-negative.

Now, from the Table of Integrals (*Gradshteyn & Ryzhik*, 2.261) we find

$$\int_{z'=0}^{z_o}\frac{\mathrm{d}z'}{\sqrt{\alpha+z'^2}} = \ln\left(z'+\sqrt{\alpha+z'^2}\right)\Big|_{z'=0}^{z_o} = \ln\left(z_o+\sqrt{\alpha+z_o^2}\right)-\ln\sqrt{\alpha}; \qquad \alpha\ge 0.$$

When $z_o\to\infty$, the first term on the right-hand-side of the above equation becomes very large, but its variations with α become insignificant. We can, therefore, drop this term, which does not vary with α, even though its magnitude is infinite. We will have

$$A_\phi(r) = -\frac{\mu_o J_{so}R}{4\pi}\int_{\phi'=0}^{2\pi}\sin\phi'\ln(R^2+r^2-2rR\sin\phi')\mathrm{d}\phi'$$

$$= -\frac{\mu_o J_{so}R}{4\pi}\left\{\int_{\phi'=0}^{2\pi}\sin\phi'\ln(R^2)\mathrm{d}\phi'+\int_{\phi'=0}^{2\pi}\sin\phi'\ln[1-2(r/R)\sin\phi'+(r/R)^2]\mathrm{d}\phi'\right\}.$$

The first integral on the right-hand-side of the above equation is zero. As for the second integral, when ϕ' is replaced by $(\frac{1}{2}\pi-\phi')$, $\sin\phi'$ changes to $\cos\phi'$, but nothing else changes because the integral covers one full period of the sine and cosine functions. We will have

$$A_\phi(r) = -\frac{\mu_o J_{so}R}{2\pi}\int_{\phi'=0}^{\pi}\cos\phi'\ln[1-2(r/R)\cos\phi'+(r/R)^2]\mathrm{d}\phi' = -\frac{\mu_o J_{so}R}{2\pi}\begin{cases}-\pi(r/R); & r\le R,\\ -\pi(R/r); & r>R.\end{cases}$$

Therefore,

$$A_\phi(r) = \begin{cases}\frac{1}{2}\mu_o J_{so}\,r; & r\le R,\\ \frac{1}{2}\mu_o J_{so}R^2/r; & r>R.\end{cases}$$