Problem 17) a) The surface current-density is readily seen to be $\boldsymbol{J}_s = (I_o/2\pi R)\hat{\boldsymbol{z}}$. In what follows, we will break up the hollow cylinder into a large number of infinitesimal straight wires, each having a width $R\Delta\phi$, all directed along z, and all having the same current density $\boldsymbol{J}_s$. The vector potential of any such wire at a radial distance ρ_o is given by

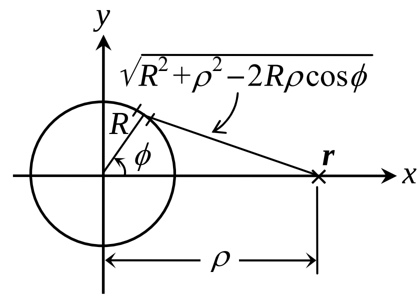
$$\boldsymbol{A}(\rho_o) = \frac{\mu_o}{4\pi} \lim_{z_o\to\infty} \int_{z'=-z_o}^{z_o} \frac{\boldsymbol{J}_s R\Delta\phi}{\sqrt{z'^2+\rho_o^2}} \mathrm{d}z' = \frac{\mu_o I_o \Delta\phi\hat{\boldsymbol{z}}}{4\pi^2} \lim_{z_o\to\infty} \int_{z'=0}^{z_o} \frac{\mathrm{d}z'}{\sqrt{z'^2+\rho_o^2}}$$

Gradshteyn & Ryzhik, 2.261

$$= \frac{\mu_o I_o \Delta\phi\hat{\boldsymbol{z}}}{4\pi^2} \lim_{z_o\to\infty} \ln\left(z' + \sqrt{z'^2+\rho_o^2}\right)\Big|_{z'=0}^{z_o} = \frac{\mu_o I_o \Delta\phi\hat{\boldsymbol{z}}}{4\pi^2}\left[\lim_{z_o\to\infty} \ln\left(z_o + \sqrt{z_o^2+\rho_o^2}\right) - \ln\rho_o\right].$$

When $z_o\to\infty$, the first term on the right-hand-side of the above equation becomes very large, but its variations with ρ_o become insignificant. We can, therefore, drop this term, which does not vary with ρ_o, even though its magnitude is infinite. We will have $\boldsymbol{A}(\rho_o) = -(\mu_o I_o \ln\rho_o \hat{\boldsymbol{z}}/4\pi^2)\Delta\phi$.

b) For a single wire located at the azimuthal angle ϕ, the distance to $\boldsymbol{r}$ is $\sqrt{R^2+\rho^2-2R\rho\cos\phi}$, as shown in the figure. The current in this wire is $(I_o/2\pi)\mathrm{d}\phi$. The corresponding vector potential at point $\boldsymbol{r}$ is thus given by



$$\boldsymbol{A}(\boldsymbol{r}) = -\frac{\mu_o I_o \hat{\boldsymbol{z}}}{4\pi^2} \ln\sqrt{R^2+\rho^2-2R\rho\cos\phi}\ \mathrm{d}\phi.$$

Integrating over all the (infinitesimal) wires around the cylinder, we obtain

$$\boldsymbol{A}(\boldsymbol{r}) = -\frac{\mu_o I_o \hat{\boldsymbol{z}}}{4\pi^2} \int_{\phi=0}^{2\pi} \ln\sqrt{R^2+\rho^2-2R\rho\cos\phi}\ \mathrm{d}\phi$$

$$= -\frac{\mu_o I_o \hat{\boldsymbol{z}}}{4\pi^2} \int_{\phi=0}^{\pi} \{\ln R^2 + \ln[1-2(\rho/R)\cos\phi + (\rho/R)^2]\}\,\mathrm{d}\phi$$

$$= -\frac{\mu_o I_o \hat{\boldsymbol{z}}}{4\pi^2}\begin{cases} 2\pi\ln R + 0 \\ 2\pi\ln R + 2\pi\ln(\rho/R)\end{cases} = \begin{cases} -\dfrac{\mu_o I_o \ln R}{2\pi}\hat{\boldsymbol{z}}; & \rho\le R, \\ -\dfrac{\mu_o I_o \ln\rho}{2\pi}\hat{\boldsymbol{z}}; & \rho > R.\end{cases}$$

c) The magnetic field is readily computed as the curl of the vector potential $\boldsymbol{A}(\boldsymbol{r})$, that is,

$$\boldsymbol{B}(\boldsymbol{r}) = \boldsymbol{\nabla}\times\boldsymbol{A}(\boldsymbol{r}) \quad\to\quad \mu_o\boldsymbol{H}(\boldsymbol{r}) = -(\partial A_z/\partial\rho)\hat{\boldsymbol{\phi}} \quad\to\quad \boldsymbol{H}(\boldsymbol{r}) = \begin{cases} 0; & \rho\le R, \\ (I_o/2\pi\rho)\hat{\boldsymbol{\phi}}; & \rho > R.\end{cases}$$

The above $\boldsymbol{H}(\boldsymbol{r})$ satisfies Ampere's law, namely, $\oint_{\text{circle}} \boldsymbol{H}\cdot\mathrm{d}\boldsymbol{\ell} = 2\pi\rho(I_o/2\pi\rho) = I_o$, for $\rho > R$. At the cylinder surface, the H-field just inside the cylinder vanishes, while the field just outside is $(I_o/2\pi R)\hat{\boldsymbol{\phi}}$. This discontinuity in the tangential H-field is precisely equal to the surface current density J_s, and perpendicular to it, confirming that the relevant boundary condition is indeed satisfied.