Problem 12)

$$\psi(\vec{r}) = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{\sqrt{x^2+y^2+(z-\frac{1}{2}d)^2}} - \frac{1}{\sqrt{x^2+y^2+(z+\frac{1}{2}d)^2}}\right)$$

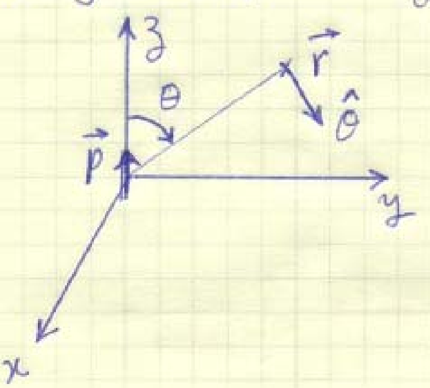
Now, $x^2+y^2+(z\pm\frac{1}{2}d)^2 = x^2+y^2+z^2+\frac{1}{4}d^2 \pm zd = r^2+\frac{d^2}{4} \pm zd$.
Since $r >> d$, we drop the term $d^2/4$ from the above expression.
We'll have:

$$\psi(\vec{r}) \cong \frac{q}{4\pi\varepsilon_0}\left[(r^2-zd)^{-1/2} - (r^2+zd)^{-1/2}\right]$$

$$= \frac{q}{4\pi\varepsilon_0 r}\left[\left(1-\frac{zd}{r^2}\right)^{-1/2} - \left(1+\frac{zd}{r^2}\right)^{-1/2}\right]$$

We now use the approximation $(1\pm\epsilon)^{-1/2} \simeq 1 \mp \frac{1}{2}\epsilon$ (See HW#1, Problem 10). Thus $\psi(\vec{r}) \cong \frac{q}{4\pi\varepsilon_0 r}\left[\left(1+\frac{zd}{2r^2}\right) - \left(1-\frac{zd}{2r^2}\right)\right] = \frac{qdz}{4\pi\varepsilon_0 r^3}$

Noting the definition of dipole moment, $\vec{p} = (qd)\hat{z}$, and the



fact that $z = \vec{r}\cdot\hat{z} = r\cos\theta$,
We can write $\psi(\vec{r}) = \dfrac{\vec{p}\cdot\vec{r}}{4\pi\varepsilon_0 r^3}$

In spherical coordinates, the unit vector $\hat{r}$ is written as $\vec{r}/r$. We may then write: $\psi(\vec{r}) = \dfrac{\vec{p}\cdot\hat{r}}{4\pi\varepsilon_0 r^2}$

Note that the potential function of a point charge is proportional to $1/r$, but the potential function for a dipole drops as $1/r^2$.

The $\vec{E}$-field is given by $\vec{E} = -\vec{\nabla}\psi$. In spherical Coordinates:

$$\vec{E}(\vec{r}) = -\frac{\partial \psi}{\partial r}\hat{r} - \frac{1}{r}\frac{\partial \psi}{\partial \theta}\hat{\theta} - \frac{1}{r \sin\theta}\frac{\partial \psi}{\partial \phi}\hat{\phi}$$

$$= -\frac{\partial}{\partial r}\left(\frac{p \cos\theta}{4\pi\epsilon_0 r^2}\right)\hat{r} - \frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{p \cos\theta}{4\pi\epsilon_0 r^2}\right)\hat{\theta}$$

$$= \frac{p \cos\theta}{2\pi\epsilon_0 r^3}\hat{r} + \frac{p \sin\theta}{4\pi\epsilon_0 r^3}\hat{\theta} = \frac{1}{4\pi\epsilon_0 r^3}\left(2p\cos\theta\,\hat{r} + p\sin\theta\,\hat{\theta}\right)$$

$$= \frac{1}{4\pi\epsilon_0 r^3}\left[3p\cos\theta\,\hat{r} - \underbrace{(p\cos\theta\,\hat{r} - p\sin\theta\,\hat{\theta})}_{=\vec{p}}\right]$$

$$= \frac{3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}}{4\pi\epsilon_0 r^3}$$