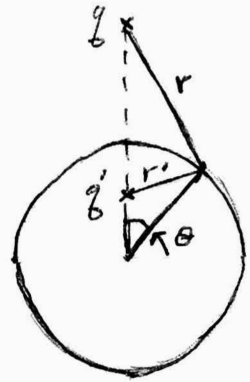
Problem 11)

a) $\Psi(r=R,\theta)=\frac{1}{4\pi\epsilon_0}\left(\frac{q}{r}+\frac{q'}{r'}\right)=0$



$$\Rightarrow \frac{q}{r}+\frac{q'}{r'}=0 \Rightarrow \frac{q}{q'}=-\frac{\sqrt{R^2+d^2-2Rd\cos\theta}}{\sqrt{R^2+d'^2-2Rd'\cos\theta}} \quad \text{for } 0\le\theta\le\pi.$$

$$\Rightarrow q^2(R^2+d'^2-2Rd'\cos\theta)=q'^2(R^2+d^2-2Rd\cos\theta)$$

Since the above equation must be valid for all θ in the interval $[0,\pi]$, the coefficients of $\cos\theta$ on both sides of the above equation must be equal. Consequently,

$$\begin{cases} 2Rd'q^2=2Rdq'^2 \\ q^2(R^2+d'^2)=q'^2(R^2+d^2) \end{cases} \Rightarrow \begin{cases} q^2/q'^2=d/d' \\ R^2=dd' \end{cases} \Rightarrow d'=\frac{R^2}{d} \text{ and } q'=-\frac{Rq}{d}$$

b) The image charge q', together with the charge q, produces the same potential at the sphere's surface (i.e., $\psi=0$). Therefore, the E-field in the region outside the sphere must be the same, whether the sphere is held at zero potential, or removed and replaced with the charge q' at distance d' from the center. Now, if a Gaussian surface is drawn immediately outside the sphere's surface, the flux of $\epsilon_0 \vec{E}$ on its surface must equal the charge on the sphere's surface. The flux of $\epsilon_0 \vec{E}$, however, is equal to q' when the sphere is replaced with the image charge. Therefore, the total charge on the sphere is $q' = -\frac{Rq}{d}$.

c) q'' must be placed at the center of the sphere in order to produce a constant potential $\psi_0 = \frac{1}{4\pi\epsilon_0}\frac{q''}{R}$ at the surface of the sphere. Therefore, $q'' = 4\pi\epsilon_0 R\psi_0$.

d) For exactly the same reasons stated in Part (b), the total charge collected on the sphere's surface will be $q'+q''$.

e) The surface charge density σ_s at each point on a conductor's surface is equal to $\epsilon_0 E_\perp$ where $E_\perp$ is the perpendicular component of the E-field immediately outside the conductor. ($E_\parallel$ is always zero for a perfect conductor.) The E-field of q and q' at the spherical surface must then be calculated in order to determine the corresponding σ_s'. For the charge q'', the surface charge density is uniform, given by $\sigma_s'' = \frac{q''}{4\pi R^2}$. This charge density must then be added to σ_s' (corresponding to the case of zero potential) to yield the total surface charge density σ_s. In the following we calculate the charge density due to q and q' only (i.e., case of $\psi_0=0$). The easiest way to determine $E_\perp$ at the surface is by way of computing $-\vec{\nabla}\psi = -\frac{\partial\psi}{\partial n}$.

where $\vec{n}$ is the normal to the sphere surface. From part (a) we write the potential $\psi(R,\theta)$ at the sphere's surface as follows:

$$\psi(R,\theta) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\sqrt{R^2+d^2-2Rd\cos\theta}} + \frac{q'}{\sqrt{R^2+d'^2-2Rd'\cos\theta}}\right)$$

Assuming that q, q', d, d' and θ are fixed, a small change in R will cause a small change in ψ, with $E_\perp = -\frac{\partial\psi}{\partial n} = -\frac{\partial\psi(R,\theta)}{\partial R}$. Therefore,

$$E_\perp = \frac{1}{4\pi\epsilon_0}\left[\frac{q(R-d\cos\theta)}{(R^2+d^2-2Rd\cos\theta)^{3/2}} + \frac{q'(R-d'\cos\theta)}{(R^2+d'^2-2Rd'\cos\theta)^{3/2}}\right].$$

Next we set $R=\sqrt{dd'}$, $q'=-Rq/d$, and use the fact that $\sigma_s(R,\theta)=\epsilon_0 E_\perp$.

$$\sigma_s(R,\theta) = \epsilon_0 E_\perp(R,\theta) = \frac{1}{4\pi}\frac{q}{(d+d'-2R\cos\theta)^{3/2}}\left[\frac{R-d\cos\theta}{d^{3/2}} - \frac{(R/d)(R-d'\cos\theta)}{d'^{3/2}}\right]$$

$$= \frac{q}{4\pi d(d+d'-2R\cos\theta)^{3/2}}\left[\sqrt{d'} - \cancel{\sqrt{d}\cos\theta} - \frac{d}{\sqrt{d'}} + \cancel{\sqrt{d}\cos\theta}\right] \Rightarrow$$

$$\sigma_s(R,\theta) = \frac{q(d'-d)}{4\pi d\sqrt{d'}(d+d'-2R\cos\theta)^{3/2}} = \frac{q(R^2-d^2)}{4\pi R(R^2+d^2-2Rd\cos\theta)^{3/2}}$$

The total charge accumulated on the sphere surface is thus given by

$$\text{Total charge} = \int_{\theta=0}^{\pi} 2\pi R^2\sin\theta\,\sigma_s(R,\theta)\,d\theta = \frac{qR^2(R^2-d^2)}{2R}\int_0^{\pi}\frac{\sin\theta}{(R^2+d^2-2Rd\cos\theta)^{3/2}}\,d\theta$$

$$= -\frac{q(R^2-d^2)}{2d}\,\frac{1}{\sqrt{R^2+d^2-2Rd\cos\theta}}\Bigg|_0^{\pi} = \frac{q(d^2-R^2)}{2d}\left(\frac{1}{d+R} - \frac{1}{d-R}\right)$$

$$\Rightarrow \text{Total charge} = -\frac{qR}{d} = q'$$