Problem 9) In problem 7, we calculated the potential function $\Psi(r)$ for a uniformly-charged hollow sphere. We now use that function to obtain the potential function for a uniformly charged solid sphere of radius R and charge density ρ_o. We must integrate the contributions of nested spherical shells having radii from $r'=0$ to $r'=R$. The charge content of each such shell is $Q(r') = (4\pi\rho_o r'^2)\,dr'$.

At a point $\vec{r}$ outside the sphere (i.e., $r \geqslant R$), we'll have:

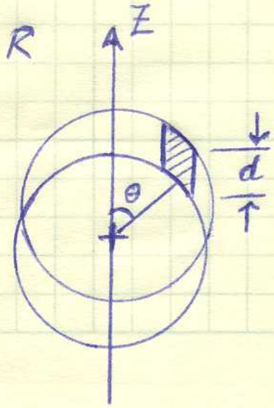
$$\Psi(r) = \frac{1}{4\pi\epsilon_o r}\int_{r'=0}^{R}(4\pi\rho_o r'^2)\,dr' = \frac{\rho_o R^3}{3\epsilon_o r}; \qquad r \geqslant R$$

At a point $\vec{r}$ inside the sphere (i.e., $r < R$), we'll have:

$$\Psi(r) = \frac{1}{4\pi\epsilon_o r}\int_{r'=0}^{r}(4\pi\rho_o r'^2)\,dr' + \frac{1}{4\pi\epsilon_o}\int_{r'=r}^{R}\frac{(4\pi\rho_o r'^2)}{r'}\,dr'$$

$$= \frac{\rho_o r^2}{3\epsilon_o} + \frac{\rho_o(R^2-r^2)}{2\epsilon_o} \Rightarrow \Psi(r) = \frac{\rho_o}{2\epsilon_o}\left(R^2 - \frac{r^2}{3}\right); \qquad r \leqslant R$$

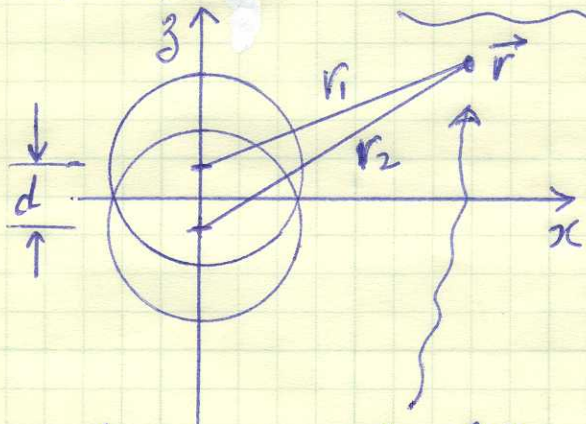
$$\vec{E}(r) = -\vec{\nabla}\Psi(r) = -\frac{\partial\Psi}{\partial r}\hat{r} = \begin{cases} +\dfrac{\rho_o r}{3\epsilon_o}\hat{r} & r \leqslant R \\ +\dfrac{\rho_o R^3}{3\epsilon_o r^2}\hat{r} & r \geqslant R \end{cases}$$



Consider two spheres of radius R, shifted by a short distance d along the Z-axis ($d \ll R$).

Next, consider a point on the surface at polar coordinates (θ, ϕ), and let a small patch of the surface area around this point be defined by small increments $\Delta\theta$, $\Delta\phi$. The area of the small patch is thus given by $\Delta S = R^2 \sin\theta\, \Delta\theta\, \Delta\phi$. With reference to the diagram at the bottom of the previous page, the area of the "charged" parallelogram is given by $(R\Delta\theta)\, d \cdot \cos\theta$. In the direction perpendicular to the plane of the paper, the thickness of this parallelogram is $R \sin\theta\, \Delta\phi$. The volume of the charged parallelpiped, therefore, is $R^2 d \sin\theta \cos\theta\, \Delta\theta\, \Delta\phi$. This volume has a charge density of ρ_0, resulting in a charge content $\Delta Q = R^2 d\, \rho_0 \sin\theta \cos\theta\, \Delta\theta\, \Delta\phi$. Dividing the charge ΔQ by the surface area ΔS of the patch yields the surface charge density $\sigma(\theta) = \rho_0 d \cos\theta$.

Comparing with problem 4, we find that $\sigma_0 = \rho_0 d$.



$$\left(\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}\right)$$

$$r_1^2 = x^2 + y^2 + (z - \tfrac{1}{2}d)^2$$

$$= x^2 + y^2 + z^2 - zd + \tfrac{1}{4}d^2$$

$$= r^2 - zd + \tfrac{1}{4}d^2 = r^2\left[1 - \left(\frac{zd}{r^2}\right) + \left(\frac{d}{2r}\right)^2\right]$$

$$\Rightarrow r_1 = r\sqrt{1 - \left(\frac{zd}{r^2}\right) + \left(\frac{d}{2r}\right)^2} \Rightarrow$$

$$r_1 \simeq r\left[1 - \frac{1}{2}\left(\frac{zd}{r^2}\right) + \frac{1}{2}\left(\frac{d}{2r}\right)^2\right] \simeq r - \frac{zd}{2r}$$

Second order in d

Similarly, $r_2^2 = r^2 + 3d + \frac{1}{4}d^2$ and $r_2 \simeq r + \frac{3d}{2r}$.

For the overlapping spheres one must add the corresponding potential functions, thus: $\hat{\psi}(\vec{r}) = \psi_1(\vec{r}) + \psi_2(\vec{r}) \Rightarrow$

$\hat{\psi}(\vec{r}) = \psi(r_1) - \psi(r_2)$. Here $\hat{\psi}(\vec{r})$ is the potential function for the overlapping spheres, and the minus sign before $\psi(r_2)$ is due to the fact that the lower sphere is negatively charged.

Therefore, for $r \leq R$, we'll have:

$$\hat{\psi}(\vec{r}) = \frac{\rho_0}{2\epsilon_0}\left(R^2 - \frac{r_1^2}{3}\right) - \frac{\rho_0}{2\epsilon_0}\left(R^2 - \frac{r_2^2}{3}\right) = \frac{\rho_0}{6\epsilon_0}(r_2^2 - r_1^2) \Rightarrow$$

$$\hat{\psi}(\vec{r}) = \frac{\rho_0}{3\epsilon_0} 3d = \frac{\rho_0 d}{3\epsilon_0} z = \frac{\sigma_0}{3\epsilon_0} z = \boxed{\frac{\sigma_0}{3\epsilon_0} r\cos\theta; \quad r \leq R}$$

For $r \geq R$ we have:

$$\hat{\psi}(\vec{r}) = \frac{\rho_0 R^3}{3\epsilon_0 r_1} - \frac{\rho_0 R^3}{3\epsilon_0 r_2} = \frac{\rho_0 R^3}{3\epsilon_0}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) = \frac{\rho_0 R^3}{3\epsilon_0}\frac{(r_2 - r_1)}{r_1 r_2}$$

$$\simeq \frac{\rho_0 R^3}{3\epsilon_0}\frac{3d/r}{r^2} = \frac{\rho_0 d R^3}{3\epsilon_0}\frac{3/r}{r^2} = \boxed{\frac{\sigma_0 R^3}{3\epsilon_0}\frac{\cos\theta}{r^2}; \quad r \geq R}$$

The above results, which are valid when $d \ll R$, are consistent with the results of problem 4.