Problem 5)

Maxwell's 4[th] equation: $\nabla\cdot\boldsymbol{B}(\boldsymbol{r})=0 \quad\rightarrow\quad \boldsymbol{B}(\boldsymbol{r})=\nabla\times\boldsymbol{A}(\boldsymbol{r})$ because $\nabla\cdot[\nabla\times\boldsymbol{A}(\boldsymbol{r})]=0$.

Maxwell's 2[nd] equation: $\nabla\times\boldsymbol{H}(\boldsymbol{r})=\boldsymbol{J}_{\text{free}}(\boldsymbol{r}) \quad\rightarrow\quad \nabla\times\boldsymbol{B}(\boldsymbol{r})=\mu_o\boldsymbol{J}_{\text{free}}(\boldsymbol{r})+\nabla\times\boldsymbol{M}(\boldsymbol{r})$

$$\rightarrow\quad \nabla\times\boldsymbol{B}(\boldsymbol{r})=\mu_o[\boldsymbol{J}_{\text{free}}(\boldsymbol{r})+\boldsymbol{J}_{\text{bound}}(\boldsymbol{r})]=\mu_o\boldsymbol{J}_{\text{total}}(\boldsymbol{r}).$$

Combining the above equations then yields $\nabla\times[\nabla\times\boldsymbol{A}(\boldsymbol{r})]=\mu_o\boldsymbol{J}_{\text{total}}(\boldsymbol{r})$. These equations involve only the transverse component of $\boldsymbol{A}(\boldsymbol{r})$, leaving its longitudinal component to be chosen freely. We thus set $\nabla\cdot\boldsymbol{A}(\boldsymbol{r})=0$, and proceed to use the definition of the Laplacian operator, $\nabla^2\boldsymbol{A}=\nabla(\nabla\cdot\boldsymbol{A})-\nabla\times(\nabla\times\boldsymbol{A})$ to arrive at $\nabla^2\boldsymbol{A}=-\mu_o\boldsymbol{J}_{\text{total}}(\boldsymbol{r})$. In the Fourier domain, this equation yields $\boldsymbol{A}(\boldsymbol{k})=\mu_o\boldsymbol{J}_{\text{total}}(\boldsymbol{k})/k^2$.

Note that our choice of gauge, $\nabla\cdot\boldsymbol{A}(\boldsymbol{r})=0$, requires that $\boldsymbol{k}\cdot\boldsymbol{A}(\boldsymbol{k})=0$, which, in turn, requires that $\boldsymbol{k}\cdot\boldsymbol{J}_{\text{total}}(\boldsymbol{k})=0$. This is obviously valid for $\boldsymbol{J}_{\text{free}}$, because the charge-current continuity equation, $\nabla\cdot\boldsymbol{J}_{\text{free}}(\boldsymbol{r})+\partial\rho_{\text{free}}(\boldsymbol{r},t)/\partial t=0$, ensures, in the absence of a time-dependent charge-density, that $\nabla\cdot\boldsymbol{J}_{\text{free}}(\boldsymbol{r})=0$, and that, consequently, $\boldsymbol{k}\cdot\boldsymbol{J}_{\text{free}}(\boldsymbol{k})=0$. The transversality requirement is also satisfied by $\boldsymbol{J}_{\text{bound}}(\boldsymbol{r})=\mu_o^{-1}\nabla\times\boldsymbol{M}(\boldsymbol{r})$, because $\boldsymbol{J}_{\text{bound}}(\boldsymbol{k})=\mathrm{i}\mu_o^{-1}\boldsymbol{k}\times\boldsymbol{M}(\boldsymbol{k})$, which yields

$$\boldsymbol{k}\cdot\boldsymbol{J}_{\text{bound}}(\boldsymbol{k})=\mathrm{i}\mu_o^{-1}\boldsymbol{k}\cdot[\boldsymbol{k}\times\boldsymbol{M}(\boldsymbol{k})]=\mathrm{i}\mu_o^{-1}(\boldsymbol{k}\times\boldsymbol{k})\cdot\boldsymbol{M}(\boldsymbol{k})=0.$$

Returning now to the vector potential $\boldsymbol{A}(\boldsymbol{r})$, recall that in Chapter 3, Problem 4(a), the 3D Fourier transform of $f(\boldsymbol{r})=1/|\boldsymbol{r}|$ was found to be $F(\boldsymbol{k})=4\pi/k^2$. We may thus write

$$\begin{aligned}
\boldsymbol{A}(\boldsymbol{r})&=\mathscr{F}^{-1}\{\boldsymbol{A}(\boldsymbol{k})\}=(2\pi)^{-3}\int_{-\infty}^{\infty}\boldsymbol{A}(\boldsymbol{k})\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})\mathrm{d}\boldsymbol{k}\\
&=(2\pi)^{-3}\int_{-\infty}^{\infty}\frac{\mu_o\boldsymbol{J}_{\text{total}}(\boldsymbol{k})\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})}{k^2}\mathrm{d}\boldsymbol{k} \qquad \boxed{\mathscr{F}\{1/|\boldsymbol{r}|\}}\\
&=\frac{\mu_o}{4\pi}(2\pi)^{-3}\int_{-\infty}^{\infty}\boldsymbol{J}_{\text{total}}(\boldsymbol{k})\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})\int_{-\infty}^{\infty}\frac{\exp(-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}'')}{|\boldsymbol{r}''|}\mathrm{d}\boldsymbol{r}''\mathrm{d}\boldsymbol{k}\\
&=\frac{\mu_o}{4\pi}(2\pi)^{-3}\int_{-\infty}^{\infty}\frac{1}{|\boldsymbol{r}''|}\int_{-\infty}^{\infty}\boldsymbol{J}_{\text{total}}(\boldsymbol{k})\exp[\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}'')]\mathrm{d}\boldsymbol{k}\,\mathrm{d}\boldsymbol{r}''\\
&=\frac{\mu_o}{4\pi}\int_{-\infty}^{\infty}\frac{\boldsymbol{J}_{\text{total}}(\boldsymbol{r}-\boldsymbol{r}'')}{|\boldsymbol{r}''|}\mathrm{d}\boldsymbol{r}''\\
&=\frac{\mu_o}{4\pi}\int_{-\infty}^{\infty}\frac{\boldsymbol{J}_{\text{total}}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}\boldsymbol{r}'. \qquad \leftarrow \boxed{\text{Change of variable: } \boldsymbol{r}'=\boldsymbol{r}-\boldsymbol{r}''}
\end{aligned}$$

The B-field may now be obtained from $\boldsymbol{B}(\boldsymbol{r})=\nabla\times\boldsymbol{A}(\boldsymbol{r})$, using the above vector potential and the identity $\nabla\times[f(\boldsymbol{r})\boldsymbol{V}(\boldsymbol{r})]=[\nabla f(\boldsymbol{r})]\times\boldsymbol{V}(\boldsymbol{r})+f(\boldsymbol{r})\nabla\times\boldsymbol{V}(\boldsymbol{r})$, as follows:

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_o}{4\pi}\int_{-\infty}^{\infty} \boldsymbol{\nabla} \times \frac{\boldsymbol{J}_{\text{total}}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}\boldsymbol{r}' = \frac{\mu_o}{4\pi}\int_{-\infty}^{\infty} \frac{\boldsymbol{J}_{\text{total}}(\boldsymbol{r}') \times (\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\mathrm{d}\boldsymbol{r}'.$$ ← Under the integral, the $\boldsymbol{\nabla}\times$ operator acts on $\boldsymbol{r}$, treating $\boldsymbol{r}'$ as a constant.

The above equation, relating a time-independent current-density distribution to its B-field, is known as the Biot-Savart law of magnetostatics. An alternative derivation relies on the fact that $\boldsymbol{B}(\boldsymbol{r})$ is a purely transverse field, as Maxwell's 4^{th} equation, $\boldsymbol{\nabla}\cdot\boldsymbol{B}(\boldsymbol{r})=0$, sets the field's longitudinal component to zero. Maxwell's 2^{nd} equation, $\boldsymbol{\nabla}\times\boldsymbol{B}(\boldsymbol{r}) = \mu_o\boldsymbol{J}_{\text{total}}(\boldsymbol{r})$, in conjunction with the knowledge that $\boldsymbol{J}_{\text{total}}(\boldsymbol{r})$ is transverse, i.e., $\boldsymbol{\nabla}\cdot\boldsymbol{J}_{\text{total}}=0$, then yields the B-field as follows:

$$\mathrm{i}\boldsymbol{k}\times\boldsymbol{B}(\boldsymbol{k}) = \mu_o\boldsymbol{J}_{\text{total}}(\boldsymbol{k}) \quad\rightarrow\quad \boldsymbol{B}(\boldsymbol{k}) = -\frac{\mathrm{i}\mu_o\boldsymbol{J}_{\text{total}}(\boldsymbol{k})\times\hat{\boldsymbol{k}}}{k}.$$

We thus have

$$\boldsymbol{B}(\boldsymbol{r}) = \mathcal{F}^{-1}\{\boldsymbol{B}(\boldsymbol{k})\} = (2\pi)^{-3}\int_{-\infty}^{\infty}\boldsymbol{B}(\boldsymbol{k})\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})\mathrm{d}\boldsymbol{k} = -\mathrm{i}(2\pi)^{-3}\int_{-\infty}^{\infty}\frac{\mu_o\boldsymbol{J}_{\text{total}}(\boldsymbol{k})\times\hat{\boldsymbol{k}}}{k}\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})\mathrm{d}\boldsymbol{k}.$$

Given that the 3D Fourier transform of the function $f(\boldsymbol{r}) = -\hat{\boldsymbol{r}}/r^2$, derived in Chapter 3, Problem 4(d), is $F(\boldsymbol{k}) = 4\pi\mathrm{i}\,\hat{\boldsymbol{k}}/k$, the preceding equation may be written

$$\boldsymbol{B}(\boldsymbol{r}) = (2\pi)^{-3}\frac{\mu_o}{4\pi}\int_{-\infty}^{\infty}\boldsymbol{J}_{\text{total}}(\boldsymbol{k})\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})\times\int_{-\infty}^{\infty}\frac{\boldsymbol{r}''\exp(-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}'')}{|\boldsymbol{r}''|^3}\mathrm{d}\boldsymbol{r}''\mathrm{d}\boldsymbol{k}$$ ← $\mathcal{F}\{\hat{\boldsymbol{r}}/r^2\} = \mathcal{F}\{\boldsymbol{r}/|\boldsymbol{r}|^3\}$

$$= (2\pi)^{-3}\frac{\mu_o}{4\pi}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty}\boldsymbol{J}_{\text{total}}(\boldsymbol{k})\exp[\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}'')]\mathrm{d}\boldsymbol{k}\right\}\times\frac{\boldsymbol{r}''}{|\boldsymbol{r}''|^3}\mathrm{d}\boldsymbol{r}''$$

$$= \frac{\mu_o}{4\pi}\int_{-\infty}^{\infty}\frac{\boldsymbol{J}_{\text{total}}(\boldsymbol{r}-\boldsymbol{r}'')\times\boldsymbol{r}''}{|\boldsymbol{r}''|^3}\mathrm{d}\boldsymbol{r}''$$

$$= \frac{\mu_o}{4\pi}\int_{-\infty}^{\infty}\frac{\boldsymbol{J}_{\text{total}}(\boldsymbol{r}')\times(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\mathrm{d}\boldsymbol{r}'.$$ ← Change of variable: $\boldsymbol{r}' = \boldsymbol{r}-\boldsymbol{r}''$

This is the same result as obtained previously by applying the curl operator to the vector potential. Either way, the B-field is computed by applying the Biot-Savart law to individual volume elements of the current-density, then integrating over the entire space. Finally, the H-field is obtained by subtracting $\boldsymbol{M}(\boldsymbol{r})$ from the above B-field, then dividing by μ_o, that is,

$$\mu_o\boldsymbol{H}(\boldsymbol{r}) = \boldsymbol{B}(\boldsymbol{r}) - \boldsymbol{M}(\boldsymbol{r}).$$
