Problem 4)

From Maxwell's 3rd equation: $\nabla\times\boldsymbol{E}(\boldsymbol{r})=0 \quad\rightarrow\quad \boldsymbol{E}(\boldsymbol{r})=-\nabla\psi(\boldsymbol{r})$ because $\nabla\times\nabla\psi(\boldsymbol{r})=0$.

From Maxwell's 1st equation: $\nabla\cdot\boldsymbol{D}(\boldsymbol{r})=\rho_{\text{free}}(\boldsymbol{r}) \quad\rightarrow\quad \varepsilon_o\nabla\cdot\boldsymbol{E}(\boldsymbol{r})=\rho_{\text{free}}(\boldsymbol{r})-\nabla\cdot\boldsymbol{P}(\boldsymbol{r})$

$$\rightarrow\quad \varepsilon_o\nabla\cdot\boldsymbol{E}(\boldsymbol{r})=\rho_{\text{free}}(\boldsymbol{r})+\rho_{\text{bound}}(\boldsymbol{r})=\rho_{\text{total}}(\boldsymbol{r}).$$

Combining the above equations then yields $\nabla^2\psi(\boldsymbol{r})=-\rho_{\text{total}}(\boldsymbol{r})/\varepsilon_o$, whose Fourier transform is $\psi(\boldsymbol{k})=\rho_{\text{total}}(\boldsymbol{k})/(\varepsilon_o k^2)$. In Chapter 3, Problem 4(a), it was found that the 3D Fourier transform of $f(\boldsymbol{r})=1/|\boldsymbol{r}|$ is $F(\boldsymbol{k})=4\pi/k^2$. We may thus write

$$\psi(\boldsymbol{r})=\mathscr{F}^{-1}\{\psi(\boldsymbol{k})\}=(2\pi)^{-3}\int_{-\infty}^{\infty}\psi(\boldsymbol{k})\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})\mathrm{d}\boldsymbol{k}$$

$$=(2\pi)^{-3}\int_{-\infty}^{\infty}\frac{\rho_{\text{total}}(\boldsymbol{k})\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})}{\varepsilon_o k^2}\mathrm{d}\boldsymbol{k} \qquad \boxed{\mathscr{F}\{1/|\boldsymbol{r}|\}}\downarrow$$

$$=(2\pi)^{-3}(4\pi\varepsilon_o)^{-1}\int_{-\infty}^{\infty}\rho_{\text{total}}(\boldsymbol{k})\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})\int_{-\infty}^{\infty}\frac{\exp(-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}'')}{|\boldsymbol{r}''|}\mathrm{d}\boldsymbol{r}''\mathrm{d}\boldsymbol{k}$$

$$=(2\pi)^{-3}(4\pi\varepsilon_o)^{-1}\int_{-\infty}^{\infty}\frac{1}{|\boldsymbol{r}''|}\int_{-\infty}^{\infty}\rho_{\text{total}}(\boldsymbol{k})\exp[\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}'')]\mathrm{d}\boldsymbol{k}\mathrm{d}\boldsymbol{r}''$$

$$=(4\pi\varepsilon_o)^{-1}\int_{-\infty}^{\infty}\frac{\rho_{\text{total}}(\boldsymbol{r}-\boldsymbol{r}'')}{|\boldsymbol{r}''|}\mathrm{d}\boldsymbol{r}''$$

$$=(4\pi\varepsilon_o)^{-1}\int_{-\infty}^{\infty}\frac{\rho_{\text{total}}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}\boldsymbol{r}'. \qquad \leftarrow \boxed{\text{Change of variable: } \boldsymbol{r}'=\boldsymbol{r}-\boldsymbol{r}''}$$

The E-field is now obtained from the above scalar potential using $\boldsymbol{E}(\boldsymbol{r})=-\nabla\psi(\boldsymbol{r})$, as follows:

$$\boldsymbol{E}(\boldsymbol{r})=-(4\pi\varepsilon_o)^{-1}\int_{-\infty}^{\infty}\rho_{\text{total}}(\boldsymbol{r}')\nabla(|\boldsymbol{r}-\boldsymbol{r}'|^{-1})\,\mathrm{d}\boldsymbol{r}'=(4\pi\varepsilon_o)^{-1}\int_{-\infty}^{\infty}\frac{\rho_{\text{total}}(\boldsymbol{r}')(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\mathrm{d}\boldsymbol{r}'. \leftarrow$$

Under the integral, the ∇ operator acts on $\boldsymbol{r}$, treating $\boldsymbol{r}'$ as a constant.

Alternatively, one could observe that Maxwell's first equation, $\varepsilon_o\nabla\cdot\boldsymbol{E}(\boldsymbol{r})=\rho_{\text{total}}(\boldsymbol{r})$, yields the longitudinal component of the E-field, as follows:

$$\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{E}(\boldsymbol{k})=\rho_{\text{total}}(\boldsymbol{k})/\varepsilon_o \quad\rightarrow\quad \boldsymbol{E}_{||}(\boldsymbol{k})=-\frac{\mathrm{i}\rho_{\text{total}}(\boldsymbol{k})\hat{\boldsymbol{k}}}{\varepsilon_o k}.$$

However, Maxwell's third equation, $\nabla\times\boldsymbol{E}(\boldsymbol{r})=0$, implies that the transverse component of $\boldsymbol{E}(\boldsymbol{r})$ is zero, and that, therefore, $\boldsymbol{E}(\boldsymbol{r})$ is purely longitudinal. Consequently, $\boldsymbol{E}(\boldsymbol{k})=\boldsymbol{E}_{||}(\boldsymbol{k})$, and we have

$$\boldsymbol{E}(\boldsymbol{r})=\mathscr{F}^{-1}\{\boldsymbol{E}(\boldsymbol{k})\}=(2\pi)^{-3}\int_{-\infty}^{\infty}\boldsymbol{E}(\boldsymbol{k})\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})\mathrm{d}\boldsymbol{k}=-\mathrm{i}(2\pi)^{-3}\int_{-\infty}^{\infty}\frac{\rho_{\text{total}}(\boldsymbol{k})\hat{\boldsymbol{k}}}{\varepsilon_o k}\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})\mathrm{d}\boldsymbol{k}.$$

Now, the 3D Fourier transform of $f(\boldsymbol{r})=-\hat{\boldsymbol{r}}/r^2$ was found in Chapter 3, Problem 4(d), to be $F(\boldsymbol{k})=4\pi\mathrm{i}\hat{\boldsymbol{k}}/k$. We may thus write

$$\mathcal{F}\{\hat{\boldsymbol{r}}/r^2\} = \mathcal{F}\{\boldsymbol{r}/|\boldsymbol{r}|^3\}$$

$$\boldsymbol{E}(\boldsymbol{r}) = (2\pi)^{-3}(4\pi\varepsilon_o)^{-1}\int_{-\infty}^{\infty} \rho_{\text{total}}(\boldsymbol{k})\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})\int_{-\infty}^{\infty} \frac{\boldsymbol{r}''\exp(-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}'')}{|\boldsymbol{r}''|^3}\mathrm{d}\boldsymbol{r}''\mathrm{d}\boldsymbol{k}$$

$$= (2\pi)^{-3}(4\pi\varepsilon_o)^{-1}\int_{-\infty}^{\infty} \frac{\boldsymbol{r}''}{|\boldsymbol{r}''|^3}\int_{-\infty}^{\infty} \rho_{\text{total}}(\boldsymbol{k})\exp[\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}'')]\mathrm{d}\boldsymbol{k}\mathrm{d}\boldsymbol{r}''$$

$$= (4\pi\varepsilon_o)^{-1}\int_{-\infty}^{\infty} \frac{\boldsymbol{r}''\rho_{\text{total}}(\boldsymbol{r}-\boldsymbol{r}'')}{|\boldsymbol{r}''|^3}\mathrm{d}\boldsymbol{r}''$$

$$= (4\pi\varepsilon_o)^{-1}\int_{-\infty}^{\infty} \frac{(\boldsymbol{r}-\boldsymbol{r}')\rho_{\text{total}}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\mathrm{d}\boldsymbol{r}'.$$

← Change of variable: $\boldsymbol{r}' = \boldsymbol{r} - \boldsymbol{r}''$

This is the same result as obtained previously by applying the gradient operator to the scalar potential. Either way, the E-field is seen to be computed by applying Coulomb's law to individual volume elements of charge, then integrating over the entire space. Finally, the D-field is obtained by adding $\boldsymbol{P}(\boldsymbol{r})$ to ε_0 times the above E-field, that is,

$$\boldsymbol{D}(\boldsymbol{r}) = \varepsilon_o\boldsymbol{E}(\boldsymbol{r}) + \boldsymbol{P}(\boldsymbol{r}).$$
