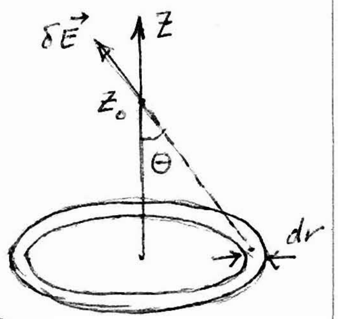
Opti 501 Solutions

Problem 3)



a) $E_z(z=z_0)=\dfrac{1}{4\pi\epsilon_0}\displaystyle\int_{r=0}^{R}\dfrac{2\pi r\sigma_s\cos\theta}{r^2+z_0^2}\,dr$

$=\dfrac{1}{4\pi\epsilon_0}\displaystyle\int_{r=0}^{R}\dfrac{2\pi r\sigma_s z_0}{(r^2+z_0^2)^{3/2}}\,dr=\dfrac{\sigma_s}{2\epsilon_0}\displaystyle\int_{x=0}^{R/z_0}\dfrac{x\,dx}{(1+x^2)^{3/2}}$

$=-\dfrac{\sigma_s}{2\epsilon_0}(1+x^2)^{-1/2}\Big|_{x=0}^{R/z_0}\Rightarrow E_z(z=z_0)=\dfrac{\sigma_s}{2\epsilon_0}\left(1-\dfrac{1}{\sqrt{1+\frac{R^2}{z_0^2}}}\right)$

b) $\Psi(z=z_0)=\dfrac{1}{4\pi\epsilon_0}\displaystyle\int_{r=0}^{R}\dfrac{2\pi r\sigma_s}{\sqrt{r^2+z_0^2}}\,dr=\dfrac{\sigma_s z_0}{2\epsilon_0}\displaystyle\int_{x=0}^{R/z_0}\dfrac{x\,dx}{\sqrt{1+x^2}}=\dfrac{\sigma_s z_0}{2\epsilon_0}\sqrt{1+x^2}\Big|_{x=0}^{R/z_0}$

$\Rightarrow \Psi(z=z_0)=\dfrac{\sigma_s}{2\epsilon_0}\left(\sqrt{R^2+z_0^2}-z_0\right)$

c) $E_z(z=z_0)=-\dfrac{\partial\Psi}{\partial z}\Big|_{z=z_0}=-\dfrac{\sigma_s}{2\epsilon_0}\left(\dfrac{z_0}{\sqrt{R^2+z_0^2}}-1\right)=\dfrac{\sigma_s}{2\epsilon_0}\left(1-\dfrac{1}{\sqrt{1+\frac{R^2}{z_0^2}}}\right)$ ✓

d) When $R\to\infty$ the second term in the expression for E_z approaches zero. We then have $E_z(z=z_0)\xrightarrow[R\to\infty]{}\dfrac{\sigma_s}{2\epsilon_0}$.

e) $\dfrac{\sigma_s}{2\epsilon_0}\left(1-\dfrac{1}{\sqrt{1+R^2/z_0^2}}\right)\geq 0.99\,\dfrac{\sigma_s}{2\epsilon_0}\Rightarrow\dfrac{1}{\sqrt{1+R^2/z_0^2}}\leq 0.01\Rightarrow 1+\dfrac{R^2}{z_0^2}\geq 10{,}000\Rightarrow$

$R\geq\sqrt{9999}\,z_0\Rightarrow R\gtrsim 100\,z_0$

f) $Q=\pi R^2\sigma_s\Rightarrow E_z(z=z_0)=\dfrac{Q}{2\pi R^2\epsilon_0}\left(1-\dfrac{1}{\sqrt{1+R^2/z_0^2}}\right)=\dfrac{Q}{2\pi\epsilon_0 R^2}\left[1-\left(1+\dfrac{R^2}{z_0^2}\right)^{-1/2}\right]$

$\simeq\dfrac{Q}{2\pi\epsilon_0\cancel{R^2}}\left(\cancel{1}-\cancel{1}+\dfrac{1}{2}\dfrac{\cancel{R^2}}{z_0^2}\right)=\dfrac{Q}{4\pi\epsilon_0 z_0^2}$ ← This is the field of a point-charge Q at a distance z_0.