Solutions **Opti 501**

Problem 2)

a) $\frac{\partial}{\partial x}\{|r-r'|^{-1}\rho(r',t-|r-r'|/c)\} = \frac{\partial}{\partial x}(|r-r'|^{-1})\rho(r',t-|r-r'|/c)$

$-\frac{1}{c}|r-r'|^{-1}\frac{\partial}{\partial x}(|r-r'|)\frac{\partial}{\partial t}\rho(r',t-|r-r'|/c)$ ← chain rule

b) $\frac{\partial^2}{\partial x^2}\{|r-r'|^{-1}\rho(r',t-|r-r'|/c)\} = \frac{\partial^2}{\partial x^2}(|r-r'|^{-1})\rho(r',t-|r-r'|/c)$

$-\frac{1}{c}\frac{\partial}{\partial x}(|r-r'|^{-1})\frac{\partial}{\partial x}(|r-r'|)\frac{\partial}{\partial t}\rho(\cdots)+\frac{1}{c}\left[\frac{\partial}{\partial x}(|r-r'|)\right]^2|r-r'|^{-2}\frac{\partial}{\partial t}\rho(\cdots)$

$-\frac{1}{c}|r-r'|^{-1}\frac{\partial^2}{\partial x^2}(|r-r'|)\frac{\partial}{\partial t}\rho(\cdots)+\frac{1}{c^2}|r-r'|^{-1}\left[\frac{\partial}{\partial x}(|r-r'|)\right]^2\frac{\partial^2}{\partial t^2}\rho(\cdots).$

Now, $\frac{\partial}{\partial x}(|\vec r-\vec r'|) = \frac{\partial}{\partial x}\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{1/2} = \frac{(x-x')}{|\vec r-\vec r'|}$, and

$\frac{\partial^2}{\partial x^2}(|\vec r-\vec r'|) = \frac{1}{|\vec r-\vec r'|}-\frac{(x-x')^2}{|\vec r-\vec r'|^3}$. Therefore,

$\frac{\partial^2}{\partial x^2}\{|r-r'|^{-1}\rho(r',t-|r-r'|/c)\} = \frac{\partial^2}{\partial x^2}(|r-r'|^{-1})\rho(r',t-|r-r'|/c)$

$+\frac{2}{c}\frac{(x-x')^2}{|r-r'|^2}|r-r'|^{-2}\frac{\partial}{\partial t}\rho(\cdots)-\frac{1}{c}|r-r'|^{-2}\frac{\partial}{\partial t}\rho(\cdots)+\frac{1}{c}\frac{(x-x')^2}{|r-r'|^4}\frac{\partial}{\partial t}\rho(\cdots)$

$+\frac{1}{c^2}\frac{(x-x')^2}{|r-r'|^3}\frac{\partial^2}{\partial t^2}\rho(\cdots).$

c) Adding the second derivatives with respect to x, y, and z yields:

$\nabla^2\left\{\frac{\rho(r',t-|r-r'|/c)}{|r-r'|}\right\} = \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\{\cdots\} = \nabla^2\left(\frac{1}{|\vec r-\vec r'|}\right)\rho(\cdots)$

$+\frac{3}{c}\cancel{\frac{(x-x')^2+(y-y')^2+(z-z')^2}{|r-r'|^4}}\frac{\partial}{\partial t}\rho(\cdots)-\cancel{\frac{3}{c}\frac{1}{|r-r'|^2}}\frac{\partial}{\partial t}\rho(\cdots)$

$+\frac{1}{c^2}\frac{(x-x')^2+(y-y')^2+(z-z')^2}{|r-r'|^3}\frac{\partial^2}{\partial t^2}\rho(\cdots) = \nabla^2\left(\frac{1}{|\vec r-\vec r'|}\right)\rho(\cdots)+\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left\{\frac{\rho(\cdots)}{|r-r'|}\right\}$

$$\Rightarrow \left(\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\left\{\frac{\rho(\vec r',t-|\vec r-\vec r'|/c)}{|\vec r-\vec r'|}\right\} = -4\pi\delta(\vec r-\vec r')\rho(r',t-|r-r'|/c)$$

$\left(\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\psi(\vec r,t) = \frac{1}{4\pi\varepsilon_0}\int_{V'}\left(\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\left\{\frac{\rho(\vec r',t-|\vec r-\vec r'|/c)}{|\vec r-\vec r'|}\right\}dv'$

$= -\frac{1}{\varepsilon_0}\int_{V'}\delta(\vec r-\vec r')\rho(\vec r',t-|\vec r-\vec r'|/c)\,dv' = -\frac{\rho(\vec r,t)}{\varepsilon_0}$ ✓