Problem 4.47)

a) $$\boldsymbol{A}(\boldsymbol{r},t) = A_0\left[\frac{\sin(k_0 r)}{(k_0 r)^2} - \frac{\cos(k_0 r)}{k_0 r}\right]\sin\theta\cos(\omega t)\,\hat{\boldsymbol{\varphi}}.$$

Using the Taylor series expansion of sine and cosine functions, we write

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots, \qquad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots.$$

Therefore, in the limit when $r \to 0$, we have

$$\frac{\sin(k_0 r)}{(k_0 r)^2} - \frac{\cos(k_0 r)}{k_0 r} = \left[\frac{1}{k_0 r} - \frac{k_0 r}{3!} + \frac{(k_0 r)^3}{5!} - \cdots\right] - \left[\frac{1}{k_0 r} - \frac{k_0 r}{2!} + \frac{(k_0 r)^3}{4!} - \cdots\right] = \frac{k_0 r}{3} - \frac{(k_0 r)^3}{30} + \cdots.$$

It is thus seen that $A_\varphi(\boldsymbol{r},t)$ approaches zero when $r \to 0$, and that, therefore, the vector potential does not have a singularity at the origin.

b) In the Lorenz gauge, $\boldsymbol{\nabla}\cdot\boldsymbol{A} + (1/c^2)\,\partial\psi/\partial t = 0$. In the present problem, since $\psi(\boldsymbol{r},t) = 0$, it is sufficient to show that $\boldsymbol{\nabla}\cdot\boldsymbol{A} = 0$. Considering that the only component of $\boldsymbol{A}(\boldsymbol{r},t)$ is A_φ, which is independent of the azimuthal angle φ, we have $\boldsymbol{\nabla}\cdot\boldsymbol{A} = (r\sin\theta)^{-1}\partial A_\varphi/\partial\varphi = 0$. The Lorenz gauge requirement is therefore satisfied.

c) $$\boldsymbol{E}(\boldsymbol{r},t) = -\boldsymbol{\nabla}\psi - \frac{\partial\boldsymbol{A}}{\partial t} = A_0\omega\left[\frac{\sin(k_0 r)}{(k_0 r)^2} - \frac{\cos(k_0 r)}{k_0 r}\right]\sin\theta\sin(\omega t)\,\hat{\boldsymbol{\varphi}}.$$

d) $$\boldsymbol{B}(\boldsymbol{r},t) = \mu_0\boldsymbol{H}(\boldsymbol{r},t) = \boldsymbol{\nabla}\times\boldsymbol{A}(\boldsymbol{r},t) = \frac{1}{r\sin\theta}\frac{\partial(\sin\theta A_\varphi)}{\partial\theta}\hat{\boldsymbol{r}} - \frac{1}{r}\frac{\partial(rA_\varphi)}{\partial r}\hat{\boldsymbol{\theta}}$$
$$= \frac{A_0}{r}\left\{\left[\frac{\sin(k_0 r)}{(k_0 r)^2} - \frac{\cos(k_0 r)}{k_0 r}\right]\left(2\cos\theta\,\hat{\boldsymbol{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}\right) - \sin(k_0 r)\sin\theta\,\hat{\boldsymbol{\theta}}\right\}\cos(\omega t).$$

Note that $\mathcal{Lim}_{\,r\to 0}\,\boldsymbol{B}(\boldsymbol{r},t) = ⅔k_0 A_0\left(\cos\theta\,\hat{\boldsymbol{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}\right)\cos(\omega t) = ⅔k_0 A_0\hat{\boldsymbol{z}}\cos(\omega t)$ is regular.

e) $$\boldsymbol{S}(\boldsymbol{r},t) = \boldsymbol{E}\times\boldsymbol{H} = \frac{(A_0\omega)^2}{2Z_0 k_0 r}\left[\frac{\sin(k_0 r)}{(k_0 r)^2} - \frac{\cos(k_0 r)}{k_0 r}\right]\left\{\left[\frac{\sin(k_0 r)}{(k_0 r)^2} - \frac{\cos(k_0 r)}{k_0 r}\right]\left(2\cos\theta\,\hat{\boldsymbol{\theta}} - \sin\theta\,\hat{\boldsymbol{r}}\right)\right.$$
$$\left. + \sin(k_0 r)\sin\theta\,\hat{\boldsymbol{r}}\right\}\sin\theta\sin(2\omega t).$$

Since the time-averaged $\boldsymbol{S}(\boldsymbol{r},t)$ is zero, the electromagnetic energy is essentially stationary.