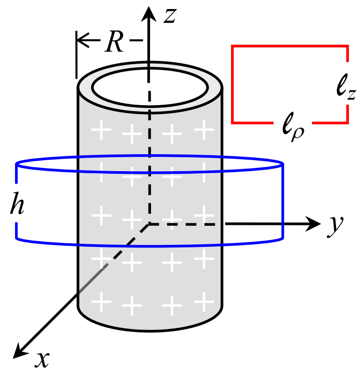
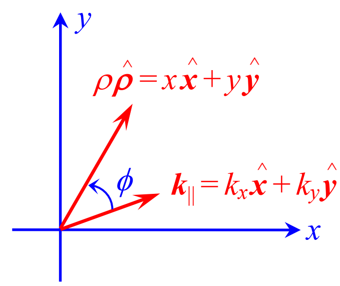


Problem 4.46) a) Because of symmetry, the E-field is independent of ϕ and z. Take a cylinder of radius ρ and height h, and write the integral form of Maxwell's first equation, $\boldsymbol{\nabla}\cdot\varepsilon_o\boldsymbol{E}=\rho_{\text{free}}$, for this cylinder. The contributions to the integral of the top and bottom surfaces of the cylinder cancel out, leaving only the contribution of the cylindrical side-wall, which is $2\pi\rho h\,\varepsilon_o E_\rho(\rho)$. Therefore, $2\pi\rho h\,\varepsilon_o E_\rho(\rho)=2\pi Rh\sigma_{so}$, where the right-hand-side of the equation gives the total electrical charge inside the cylinder of radius ρ, provided, of course, that $\rho>R$. Consequently, $E_\rho(\rho)=R\sigma_{so}/(\rho\,\varepsilon_o)$ when $\rho>R$, and $E_\rho(\rho)=0$ when $\rho<R$. From Maxwell's 3rd equation, $\boldsymbol{\nabla}\times\boldsymbol{E}=0$, we conclude that $E_\phi=0$, otherwise a circular loop of radius ρ, parallel to the xy-plane and centered on the z-axis, will have a nonzero line integral. As for E_z, consider the rectangular loop $\ell_\rho\times\ell_z$ shown in the figure. The contributions of ℓ_ρ to the line-integral of the E-field around the loop cancel out because E_ρ is independent of z. For the contributions of the vertical legs, ℓ_z, to also cancel out, it is necessary for E_z to be independent of ρ. We thus see that E_z must be constant through the entire space. In fact, because of the system's up-down symmetry, it is not difficult to see that E_z must be identically zero everywhere: There is as much reason for the E-field to point up as there is for it to point down. Therefore $E_z=0$ and we have

$$\boldsymbol{E}(\boldsymbol{r},t)=\begin{cases}(R\sigma_{so}/\varepsilon_o\rho)\hat{\boldsymbol{\rho}}; & \rho>R,\\ 0; & \rho<R.\end{cases}$$

b) The Fourier transform of the electric charge-density, $\rho(\boldsymbol{r},t)=\sigma_{so}\delta(\rho-R)$, is given by

$$\begin{aligned}\rho(\boldsymbol{k},\omega)&=\int_{-\infty}^{\infty}\sigma_{so}\delta(\rho-R)\exp[-\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\mathrm{d}\boldsymbol{r}\,\mathrm{d}t\\ &=(2\pi)^2\sigma_{so}\,\delta(k_z)\,\delta(\omega)\int_{\rho=0}^{\infty}\int_{\phi=0}^{2\pi}\delta(\rho-R)\exp(-\mathrm{i}\,k_\parallel\rho\cos\phi)\rho\,\mathrm{d}\rho\,\mathrm{d}\phi\\ &=(2\pi)^2R\sigma_{so}\,\delta(k_z)\,\delta(\omega)\int_{\phi=0}^{2\pi}\exp(-\mathrm{i}\,k_\parallel R\cos\phi)\,\mathrm{d}\phi\\ &=(2\pi)^3R\sigma_{so}\,\delta(k_z)\,\delta(\omega)\,J_0(k_\parallel R).\end{aligned}$$



The E-field is thus obtained as follows:

$$\boldsymbol{E}(\boldsymbol{r},t)=-\boldsymbol{\nabla}\psi(\boldsymbol{r},t)=-(2\pi)^{-4}\int_{-\infty}^{\infty}\frac{\mathrm{i}\boldsymbol{k}\rho(\boldsymbol{k},\omega)}{\varepsilon_o[k^2-(\omega/c)^2]}\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\mathrm{d}\boldsymbol{k}\,\mathrm{d}\omega$$

$$=-\frac{\mathrm{i}R\sigma_{so}}{2\pi\varepsilon_o}\int_{-\infty}^{\infty}\frac{\boldsymbol{k}\,\delta(k_z)\,\delta(\omega)\,J_0(k_\parallel R)}{k^2-(\omega/c)^2}\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\mathrm{d}\boldsymbol{k}\,\mathrm{d}\omega$$

$$= -\frac{\mathrm{i}\,R\sigma_{so}}{2\pi\varepsilon_{\mathrm{o}}}\int_{k_{\|}=0}^{\infty}\int_{\phi=0}^{2\pi}\frac{k_{\|}\cos\phi\,\hat{\boldsymbol{\rho}}\,J_0(k_{\|}R)}{k_{\|}^2}\exp(\mathrm{i}k_{\|}\rho\cos\phi)\,k_{\|}\mathrm{d}k_{\|}\mathrm{d}\phi$$

$$= -\frac{\mathrm{i}\,R\sigma_{so}\hat{\boldsymbol{\rho}}}{2\pi\varepsilon_{\mathrm{o}}}\int_{k_{\|}=0}^{\infty}J_0(k_{\|}R)\int_{\phi=0}^{2\pi}\cos\phi\,\exp(\mathrm{i}k_{\|}\rho\cos\phi)\,\mathrm{d}\phi\mathrm{d}k_{\|}$$

$$= \frac{R\sigma_{so}\hat{\boldsymbol{\rho}}}{\varepsilon_{\mathrm{o}}}\int_0^{\infty}J_0(k_{\|}R)\,J_1(k_{\|}\rho)\,\mathrm{d}k_{\|} \quad = \begin{cases}(R\sigma_{so}/\varepsilon_{\mathrm{o}}\rho)\hat{\boldsymbol{\rho}}; & \rho > R,\\ 0; & \rho < R.\end{cases}$$
