Problem 4.43) a) The charge-current continuity equation yields

$$\boldsymbol{\nabla}\cdot\boldsymbol{J} = \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z} = -\frac{\partial \rho}{\partial t} = \omega_0\rho_0\cos(k_0x)\sin(\omega_0 t).$$

It is sufficient to assume that $J_y = J_z = 0$, and that J_x is a function only of x and t. We will have

$$\frac{\partial}{\partial x}J_x(x,t) = \omega_0\rho_0\cos(k_0x)\sin(\omega_0 t) \quad \rightarrow \quad \boldsymbol{J}(\boldsymbol{r},t) = J_x\hat{\boldsymbol{x}} = \left(\frac{\omega_0\rho_0}{k_0}\right)\sin(k_0x)\sin(\omega_0 t)\,\hat{\boldsymbol{x}}.$$

b) $\rho(\boldsymbol{r},t)$ and $\boldsymbol{J}(\boldsymbol{r},t)$ may be expressed as superpositions of plane-waves, as follows:

$$\begin{aligned}\rho(\boldsymbol{r},t) &= \tfrac{1}{4}\rho_0[\exp(\mathrm{i}k_0x)+\exp(-\mathrm{i}k_0x)][\exp(\mathrm{i}\omega_0t)+\exp(-\mathrm{i}\omega_0t)]\\ &= \tfrac{1}{4}\rho_0\exp[\mathrm{i}(k_0x+\omega_0t)] + \tfrac{1}{4}\rho_0\exp[-\mathrm{i}(k_0x+\omega_0t)]\\ &\quad +\tfrac{1}{4}\rho_0\exp[\mathrm{i}(k_0x-\omega_0t)] + \tfrac{1}{4}\rho_0\exp[-\mathrm{i}(k_0x-\omega_0t)].\end{aligned}$$

$$\begin{aligned}\boldsymbol{J}(\boldsymbol{r},t) &= -\left(\frac{\omega_0\rho_0}{4k_0}\right)[\exp(\mathrm{i}k_0x)-\exp(-\mathrm{i}k_0x)][\exp(\mathrm{i}\omega_0t)-\exp(-\mathrm{i}\omega_0t)]\hat{\boldsymbol{x}}\\ &= -\left(\frac{\omega_0\rho_0}{4k_0}\right)\exp[\mathrm{i}(k_0x+\omega_0t)]\,\hat{\boldsymbol{x}} - \left(\frac{\omega_0\rho_0}{4k_0}\right)\exp[-\mathrm{i}(k_0x+\omega_0t)]\,\hat{\boldsymbol{x}}\\ &\quad + \left(\frac{\omega_0\rho_0}{4k_0}\right)\exp[\mathrm{i}(k_0x-\omega_0t)]\,\hat{\boldsymbol{x}} + \left(\frac{\omega_0\rho_0}{4k_0}\right)\exp[-\mathrm{i}(k_0x-\omega_0t)]\,\hat{\boldsymbol{x}}.\end{aligned}$$

All the above plane-waves have $k_x = \pm k_0$, $k_y = k_z = 0$, and $\omega = \pm\omega_0$. Therefore, the value of $[k^2-(\omega/c)^2]$ for all these plane-waves is the same, namely, $[k_0^2-(\omega_0/c)^2]$. The scalar potential for each charge-density plane-wave is obtained by multiplying the corresponding charge-density by $\frac{1}{\varepsilon_0[k_0^2-(\omega_0/c)^2]}$. Similarly, the vector potential for each current-density plane-wave is obtained by multiplying the corresponding current-density by $\frac{\mu_0}{k_0^2-(\omega_0/c)^2}$. It is readily observed that the scalar potential and vector potential plane-waves combine once again to form simple sine and cosine functions, as follows:

$$\psi(\boldsymbol{r},t) = \frac{\rho_0\cos(k_0x)\cos(\omega_0t)}{\varepsilon_0[k_0^2-(\omega_0/c)^2]},$$

$$\boldsymbol{A}(\boldsymbol{r},t) = \frac{(\mu_0\omega_0\rho_0/k_0)\sin(k_0x)\sin(\omega_0t)}{k_0^2-(\omega_0/c)^2}\hat{\boldsymbol{x}}.$$

c)
$$\begin{aligned}\boldsymbol{E}(\boldsymbol{r},t) &= -\boldsymbol{\nabla}\psi - \frac{\partial\boldsymbol{A}}{\partial t}\\ &= \frac{\rho_0k_0\sin(k_0x)\cos(\omega_0t)}{\varepsilon_0[k_0^2-(\omega_0/c)^2]}\hat{\boldsymbol{x}} - \frac{(\mu_0\omega_0^2\rho_0/k_0)\sin(k_0x)\cos(\omega_0t)}{k_0^2-(\omega_0/c)^2}\hat{\boldsymbol{x}}\\ &= \left(\frac{\rho_0}{\varepsilon_0k_0}\right)\sin(k_0x)\cos(\omega_0t)\,\hat{\boldsymbol{x}}.\end{aligned}$$

$$\boldsymbol{B}(\boldsymbol{r},t) = \boldsymbol{\nabla}\times\boldsymbol{A} = (\partial A_x/\partial z)\hat{\boldsymbol{y}} - (\partial A_x/\partial y)\hat{\boldsymbol{z}} = 0.$$

It is interesting to note that neither the current $\boldsymbol{J}(\boldsymbol{r},t)$ nor the time-dependent E-field in the present problem give rise to a magnetic field. In fact, a quick check of Maxwell's second equation reveals that $\boldsymbol{J}(\boldsymbol{r},t)$ and $\partial\boldsymbol{D}/\partial t$ exactly cancel out. The satisfaction of the remaining Maxwell's equations may also be readily verified.