Problem 4.42)

a) For the plane-wave, the Lorenz gauge formula $\boldsymbol{\nabla}\cdot\boldsymbol{A}+\frac{\partial\psi}{c^2\partial t}=0$ becomes $\boldsymbol{k}\cdot\boldsymbol{A}_0=(\omega/c^2)\psi_0$.

b)
$$\boldsymbol{E}(\boldsymbol{r},t)=-\boldsymbol{\nabla}\psi-\partial\boldsymbol{A}/\partial t=(-\mathrm{i}\boldsymbol{k}\psi_0+\mathrm{i}\omega\boldsymbol{A}_0)\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)],$$
$$\boldsymbol{B}(\boldsymbol{r},t)=\boldsymbol{\nabla}\times\boldsymbol{A}=\mathrm{i}\boldsymbol{k}\times\boldsymbol{A}_0\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)].$$

c) i) Maxwell's first equation (in free space): $\boldsymbol{\nabla}\cdot\boldsymbol{E}=0 \;\rightarrow\; \boldsymbol{k}\cdot\boldsymbol{E}=0 \;\rightarrow\; k^2\psi_0-\omega\boldsymbol{k}\cdot\boldsymbol{A}_0=0$. This result may now be combined with that obtained in part (a) to yield

$$[k^2-(\omega/c)^2]\psi_0=0.$$

If $\psi_0\neq 0$, we must have $k^2=(\omega/c)^2$.

ii) Maxwell's second equation (in free space):

$$\boldsymbol{\nabla}\times\boldsymbol{B}=\mu_0\varepsilon_0\,\partial\boldsymbol{E}/\partial t \;\rightarrow\; \mathrm{i}^2\boldsymbol{k}\times(\boldsymbol{k}\times\boldsymbol{A}_0)=-\mathrm{i}(\omega/c^2)(-\mathrm{i}\boldsymbol{k}\psi_0+\mathrm{i}\omega\boldsymbol{A}_0)$$
$$\rightarrow\; (\boldsymbol{k}\cdot\boldsymbol{A}_0)\boldsymbol{k}-k^2\boldsymbol{A}_0=(\omega/c^2)(\boldsymbol{k}\psi_0-\omega\boldsymbol{A}_0)$$
$$\rightarrow\; [\boldsymbol{k}\cdot\boldsymbol{A}_0-(\omega/c^2)\psi_0]\boldsymbol{k}=[k^2-(\omega/c)^2]\boldsymbol{A}_0$$

The preceding equation, when combined with the Lorenz gauge result obtained in part (a), yields

$$[k^2-(\omega/c)^2]\boldsymbol{A}_0=0,$$

which gives a non-zero value for $\boldsymbol{A}_0$ only if $k^2=(\omega/c)^2$.

iii) Maxwell's third equation:

$$\boldsymbol{\nabla}\times\boldsymbol{E}=-\,\partial\boldsymbol{B}/\partial t \;\rightarrow\; \mathrm{i}\boldsymbol{k}\times(-\mathrm{i}\boldsymbol{k}\psi_0+\mathrm{i}\omega\boldsymbol{A}_0)=\mathrm{i}^2\omega\boldsymbol{k}\times\boldsymbol{A}_0$$
$$\rightarrow\; -\underbrace{(\boldsymbol{k}\times\boldsymbol{k})}_{0}\psi_0+\omega\boldsymbol{k}\times\boldsymbol{A}_0=\omega\boldsymbol{k}\times\boldsymbol{A}_0 \quad \text{(automatically satisfied).}$$

iv) Maxwell's fourth equation:

$$\boldsymbol{\nabla}\cdot\boldsymbol{B}=0 \;\rightarrow\; \mathrm{i}^2\boldsymbol{k}\cdot(\boldsymbol{k}\times\boldsymbol{A}_0)=0 \;\rightarrow\; \overbrace{(\boldsymbol{k}\times\boldsymbol{k})}^{0}\cdot\boldsymbol{A}_0=0 \quad \text{(automatically satisfied).}$$