Problem 4.40)

a) $$\boldsymbol{E}(\boldsymbol{r}) = -\boldsymbol{\nabla}\psi(\boldsymbol{r}) = -\frac{\partial\psi}{\partial r}\hat{\boldsymbol{r}} - \frac{\partial\psi}{r\partial\theta}\hat{\boldsymbol{\theta}} = \begin{cases} -(P_0/3\varepsilon_0)\,(\cos\theta\,\hat{\boldsymbol{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}); & r < R, \\ (P_0R^3/3\varepsilon_0)\,(2\cos\theta\,\hat{\boldsymbol{r}} + \sin\theta\,\hat{\boldsymbol{\theta}})/r^3\,; & r \ge R. \end{cases}$$

The field inside the sphere may be further simplified and written as $\boldsymbol{E}(\boldsymbol{r}) = -(P_0/3\varepsilon_0)\hat{\boldsymbol{z}}$.

b) $$\rho^{(e)}_{\text{bound}}(\boldsymbol{r}) = -\boldsymbol{\nabla}\cdot\boldsymbol{P}(\boldsymbol{r}) = -\boldsymbol{\nabla}\cdot\left[P_0\,\text{Sphere}(r/R)(\cos\theta\,\hat{\boldsymbol{r}} - \sin\theta\,\hat{\boldsymbol{\theta}})\right]$$

$$= -\frac{\partial(r^2P_r)}{r^2\partial r} - \frac{\partial(\sin\theta P_\theta)}{r\sin\theta\partial\theta}$$

$$= -\left(\frac{P_0\cos\theta}{r^2}\right)\frac{\partial\left[r^2\,\text{Sphere}(r/R)\right]}{\partial r} + \left[\frac{P_0\text{Sphere}(r/R)}{r\sin\theta}\right]\frac{\partial\sin^2\theta}{\partial\theta}$$

$$= -\frac{P_0\cos\theta\left[2r\,\text{Sphere}(r/R) - r^2\,\delta(r-R)\right]}{r^2} + \frac{2P_0\text{Sphere}(r/R)\cos\theta}{r} = P_0\delta(r-R)\cos\theta.$$

Because of the δ-function appearing in the above charge-density profile, we can state that the sphere has a bound *surface-charge-density* $\sigma_s(r = R, \theta, \phi) = P_0\cos\theta$. Note that the surface-charge-density is positive on the upper hemisphere and negative on the lower hemisphere.

c) The parallel (or tangential) component of the E-field at the surface of the sphere is E_θ, which is found in (a) to be equal to $(P_0/3\varepsilon_0)\sin\theta$ immediately inside as well as immediately outside the sphere. The continuity requirement for the tangential E-field is, therefore, satisfied.

The perpendicular component of the D-field inside the sphere is given by

$$D_\perp = \varepsilon_0 E_r + P_r = -⅓P_0\cos\theta + P_0\cos\theta = ⅔P_0\cos\theta.$$

Outside the sphere and immediately above the surface, we have $D_\perp = \varepsilon_0 E_r = ⅔P_0\cos\theta$. Therefore, in the absence of free surface-charge-density, the continuity of $D_\perp$ is confirmed.

If, instead of $D_\perp$, we examine the perpendicular component of the E-field, we find, at the surface of the sphere, the following discontinuity in $E_\perp$:

$$E_r(r = R^+, \theta, \phi) - E_r(r = R^-, \theta, \phi) = (2P_0/3\varepsilon_0)\cos\theta - (-P_0/3\varepsilon_0)\cos\theta$$

$$= (P_0/\varepsilon_0)\cos\theta.$$

This, however, is precisely equal to the bound surface-charge-density $\sigma_s = P_0\cos\theta$ found in part (b), divided by ε_0, which is, once again, consistent with the boundary condition derived from Maxwell's 1st equation, $\boldsymbol{\nabla}\cdot\boldsymbol{E}(\boldsymbol{r},t) = \varepsilon_0^{-1}\rho^{(e)}_{\text{total}}(\boldsymbol{r},t)$.