Problem 4-39) a) The total charge Q and the magnetic dipole moment m_z are readily found as follows:

$$Q = 4\pi R^2 \sigma_{so}. \tag{1}$$

$$m_z = \mu_o \int_0^\pi \pi (R\sin\theta)^2 (\sigma_{so}\omega_o R\sin\theta) R\mathrm{d}\theta = \pi R^4 \mu_o \sigma_{so}\omega_o \int_0^\pi \sin^3\theta\,\mathrm{d}\theta = (\tfrac{4}{3}\pi R^3)\mu_o \sigma_{so} R\omega_o. \tag{2}$$

b) The Fourier transform of the charge-density distribution is given by

$$\rho(\boldsymbol{k}) = \int_{-\infty}^{\infty} \sigma_{so}\delta(r-R)\exp(-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})\mathrm{d}\boldsymbol{r} = \int_{r=0}^{\infty}\int_{\theta=0}^{\pi} \sigma_{so}\delta(r-R)\exp(-\mathrm{i}kr\cos\theta)2\pi r^2 \sin\theta \mathrm{d}r\mathrm{d}\theta$$

$$= 2\pi R^2\sigma_{so}\int_{\theta=0}^{\pi}\sin\theta\exp(-\mathrm{i}kR\cos\theta)\mathrm{d}\theta = \frac{2\pi R^2\sigma_{so}}{\mathrm{i}kR}\exp(-\mathrm{i}kR\cos\theta)\Big|_{\theta=0}^{\pi} = \frac{4\pi R\sigma_{so}\sin(kR)}{k}. \tag{3}$$

The electric field may now be obtained from $\boldsymbol{E}(\boldsymbol{r}) = -\boldsymbol{\nabla}\psi(\boldsymbol{r})$, as follows:

$$\boldsymbol{E}(\boldsymbol{r}) = -\frac{1}{(2\pi)^3}\int \frac{\mathrm{i}\boldsymbol{k}\rho(\boldsymbol{k})}{\varepsilon_o k^2}\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})\mathrm{d}\boldsymbol{k}$$

$$= -\frac{\mathrm{i}4\pi R\sigma_{so}\hat{\boldsymbol{r}}}{(2\pi)^3\varepsilon_o}\int_{k=0}^{\infty}\int_{\theta=0}^{\pi}\frac{\sin(kR)}{k^2}\cos\theta\exp(\mathrm{i}kr\cos\theta)2\pi k^2\sin\theta\mathrm{d}k\mathrm{d}\theta$$

$$= -\frac{\mathrm{i}R\sigma_{so}\hat{\boldsymbol{r}}}{\pi\varepsilon_o}\int_{k=0}^{\infty}\sin(kR)\int_{\theta=0}^{\pi}\sin\theta\cos\theta\exp(\mathrm{i}kr\cos\theta)\mathrm{d}\theta\mathrm{d}k \quad \leftarrow \text{(G\&R 3.715-11)}$$

$$= -\frac{\mathrm{i}R\sigma_{so}\hat{\boldsymbol{r}}}{\pi\varepsilon_o}\int_0^{\infty}\sin(kR)\frac{2\mathrm{i}(\sin kr - kr\cos kr)}{(kr)^2}\mathrm{d}k$$

$$= \frac{2R\sigma_{so}\hat{\boldsymbol{r}}}{\pi\varepsilon_o}\left\{\frac{1}{r^2}\int_0^{\infty}\frac{\sin(kR)\sin(kr)}{k^2}\mathrm{d}k - \frac{1}{r}\int_0^{\infty}\frac{\sin(kR)\cos(kr)}{k}\mathrm{d}k\right\} \quad \leftarrow \text{(G\&R 3.741-2,3)}$$

$$= \begin{cases}\dfrac{\sigma_{so}R^2}{\varepsilon_o r^2}\hat{\boldsymbol{r}}; & r > R\\ 0; & r < R\end{cases} = \begin{cases}\dfrac{Q}{4\pi\varepsilon_o r^2}\hat{\boldsymbol{r}}; & r > R,\\ 0; & r < R.\end{cases} \tag{4}$$

The Fourier transform of the current-density distribution is given by

$$\boldsymbol{\mathcal{J}}(\boldsymbol{k}) = \int_{-\infty}^{\infty}\sigma_{so}R\omega_o\delta(r-R)(\hat{\boldsymbol{z}}\times\hat{\boldsymbol{r}})\exp(-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})\mathrm{d}\boldsymbol{r}$$

$$= \sigma_{so}R\omega_o(\hat{\boldsymbol{z}}\times\hat{\boldsymbol{k}})\int_{r=0}^{\infty}\int_{\theta=0}^{\pi}\delta(r-R)\cos\theta\exp(-\mathrm{i}kr\cos\theta)2\pi r^2\sin\theta\mathrm{d}r\mathrm{d}\theta$$

$$= 2\pi\sigma_{so}R^3\omega_o(\hat{\boldsymbol{z}}\times\hat{\boldsymbol{k}})\int_{\theta=0}^{\pi}\sin\theta\cos\theta\exp(-\mathrm{i}kR\cos\theta)\mathrm{d}\theta \quad \leftarrow \text{(G\&R 3.715-11)}$$

$$= -\mathrm{i}4\pi R\omega_o\sigma_{so}\frac{\sin(kR) - kR\cos(kR)}{k^2}(\hat{\boldsymbol{z}}\times\hat{\boldsymbol{k}}). \quad \leftarrow \textbf{(Note: } \hat{\boldsymbol{z}}\times\hat{\boldsymbol{k}} = \sin\theta\,\hat{\boldsymbol{\phi}}_k\textbf{)} \tag{5}$$

Having determined $\boldsymbol{\mathcal{J}}(\boldsymbol{k})$, we now proceed to calculate the vector potential $\boldsymbol{A}(\boldsymbol{r})$, as follows:

$$
\begin{aligned}
\boldsymbol{A}(\boldsymbol{r}) &= \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}\frac{\mu_o \mathcal{J}(\boldsymbol{k})}{k^2}\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})\mathrm{d}\boldsymbol{k} \\
&= -\frac{\mathrm{i}4\pi\mu_o R\omega_o\sigma_{so}}{(2\pi)^3}(\hat{\boldsymbol{z}}\times\hat{\boldsymbol{r}})\int_{k=0}^{\infty}\int_{\theta=0}^{\pi}\frac{\sin(kR)-kR\cos(kR)}{k^4}\cos\theta\exp(\mathrm{i}kr\cos\theta)2\pi k^2\sin\theta\mathrm{d}k\mathrm{d}\theta \\
&= -\frac{\mathrm{i}\mu_o R\omega_o\sigma_{so}}{\pi}(\hat{\boldsymbol{z}}\times\hat{\boldsymbol{r}})\int_{k=0}^{\infty}\frac{\sin(kR)-kR\cos(kR)}{k^2}\int_{\theta=0}^{\pi}\sin\theta\cos\theta\exp(\mathrm{i}kr\cos\theta)\mathrm{d}\theta\mathrm{d}k \\
&= \frac{2\mu_o R\omega_o\sigma_{so}}{\pi r^2}\sin\theta\hat{\boldsymbol{\phi}}\int_{k=0}^{\infty}\frac{[\sin(kR)-kR\cos(kR)][\sin(kr)-kr\cos(kr)]}{k^4}\mathrm{d}k \\
&= \begin{cases}\frac{1}{3}(\mu_o R^4\omega_o\sigma_{so})\, r^{-2}\sin\theta\hat{\boldsymbol{\phi}}; & r>R, \\ \frac{1}{3}(\mu_o R\,\omega_o\sigma_{so})\, r\sin\theta\hat{\boldsymbol{\phi}}; & r<R.\end{cases}
\end{aligned} \tag{6}
$$

The magnetic field is now obtained as follows:

$$
\begin{aligned}
\boldsymbol{B}(\boldsymbol{r}) = \boldsymbol{\nabla}\times\boldsymbol{A}(\boldsymbol{r}) &= \begin{cases}\frac{1}{3}(\mu_o R^4\omega_o\sigma_{so})\, r^{-3}(2\cos\theta\hat{\boldsymbol{r}}+\sin\theta\hat{\boldsymbol{\theta}}); & r>R \\ \frac{2}{3}(\mu_o R\,\omega_o\sigma_{so})(\cos\theta\hat{\boldsymbol{r}}-\sin\theta\hat{\boldsymbol{\theta}}); & r<R\end{cases} \\
&= \begin{cases}\dfrac{m_z(2\cos\theta\hat{\boldsymbol{r}}+\sin\theta\hat{\boldsymbol{\theta}})}{4\pi r^3}; & r>R, \\ \dfrac{2}{3}\dfrac{m_z}{(4\pi R^3/3)}\hat{\boldsymbol{z}}; & r<R.\end{cases}
\end{aligned} \tag{7}
$$

c) The E-field energy of the spherical shell is obtained by integration over the field intensity outside the shell, as the field inside is zero.

$$
\mathcal{E}_E = \int_{r=R}^{\infty}\frac{1}{2}\varepsilon_o\left(\frac{Q}{4\pi\varepsilon_o r^2}\right)^2 4\pi r^2\mathrm{d}r = \frac{Q^2}{8\pi\,\varepsilon_o R}. \tag{8}
$$

The H-field energy has contributions from the magnetic field inside as well as outside the shell, namely,

$$
\begin{aligned}
\mathcal{E}_H &= \frac{1}{2}\mu_o\left|\frac{2m_z}{3\mu_o(4\pi R^3/3)}\hat{\boldsymbol{z}}\right|^2(4\pi R^3/3) + \int_{r=R}^{\infty}\int_{\theta=0}^{\pi}\frac{1}{2}\mu_o\left|\frac{m_z(2\cos\theta\hat{\boldsymbol{r}}+\sin\theta\hat{\boldsymbol{\theta}})}{4\mu_o\pi r^3}\right|^2 2\pi r^2\sin\theta\mathrm{d}r\mathrm{d}\theta \\
&= \frac{m_z^2}{6\mu_o\pi R^3}+\frac{m_z^2}{16\pi\mu_o}\int_{r=R}^{\infty}r^{-4}\mathrm{d}r\int_{\theta=0}^{\pi}(4\sin\theta-3\sin^3\theta)\,\mathrm{d}\theta = \frac{m_z^2}{6\pi\mu_o R^3}+\frac{m_z^2}{12\pi\mu_o R^3} = \frac{m_z^2}{4\pi\mu_o R^3}.
\end{aligned} \tag{9}
$$

The H-field energy is seen to be divided between inside and outside the sphere, with the inside field containing twice as much energy as the outside field.

Inside the sphere, the Poynting vector is zero (because the E-field is zero), but outside the sphere it is given by

$$
\boldsymbol{S}(\boldsymbol{r}) = \boldsymbol{E}(\boldsymbol{r})\times\boldsymbol{H}(\boldsymbol{r}) = \frac{Q}{4\pi\varepsilon_o r^2}\hat{\boldsymbol{r}}\times\frac{m_z(2\cos\theta\hat{\boldsymbol{r}}+\sin\theta\hat{\boldsymbol{\theta}})}{4\pi\mu_o r^3} = \frac{Qm_z c^2\sin\theta\hat{\boldsymbol{\phi}}}{16\pi^2 r^5}; \qquad r>R. \tag{10}
$$

The EM angular momentum density with respect to the origin is given by $\boldsymbol{L}(\boldsymbol{r}) = \boldsymbol{r} \times \boldsymbol{S}(\boldsymbol{r})/c^2$; therefore, the total angular momentum of the spinning sphere may be obtained as follows:

$$\boldsymbol{\mathcal{L}} = \int_{-\infty}^{\infty} \boldsymbol{L}(\boldsymbol{r})\mathrm{d}\boldsymbol{r} = \iiint\limits_{\substack{\text{outside}\\ \text{sphere}}} \boldsymbol{r} \times \frac{Qm_z \sin\theta \hat{\boldsymbol{\phi}}}{16\pi^2 r^5} \mathrm{d}\boldsymbol{r} = \frac{Qm_z \hat{\boldsymbol{z}}}{8\pi} \int_{r=R}^{\infty} r^{-2}\mathrm{d}r \int_{\theta=0}^{\pi} \sin^3\theta \,\mathrm{d}\theta = \frac{Qm_z}{6\pi R}\hat{\boldsymbol{z}}. \qquad (11)$$

Note that this angular momentum is purely due to the electromagnetic field; as such, it is independent of the mass of the sphere.