Problem 4-38) a) With reference to problem 4-12, The E-field of the spherical dipole is given by

$$\boldsymbol{E}(\boldsymbol{r}) = \begin{cases} \dfrac{P_o R^3 (2\cos\theta\hat{\boldsymbol{r}} + \sin\theta\hat{\boldsymbol{\theta}})}{3\varepsilon_o r^3}; & r > R \\ -\dfrac{P_o \hat{\boldsymbol{z}}}{3\varepsilon_o}; & r < R \end{cases}$$

The linear momentum density is $\boldsymbol{E}(\boldsymbol{r})\times\boldsymbol{H}(\boldsymbol{r})/c^2$. Since $\boldsymbol{H}(\boldsymbol{r}) = J_{so}\hat{\boldsymbol{y}}$ is constant within the gap and zero outside, the total momentum may be found by integrating the E-field throughout the gap region, then cross-multiplying it into $(J_{so}/c^2)\hat{\boldsymbol{y}}$. Due to symmetry, the x-component of the E-field integrates to zero, leaving only the z-component to evaluate. We have

$$\int_{\text{gap}} E_z(\boldsymbol{r})\mathrm{d}\boldsymbol{r} = -\frac{(4\pi R^3/3)P_o}{3\varepsilon_o} + \frac{P_o R^3}{3\varepsilon_o}\Big\{\int_{r=R}^{d/2}\int_{\theta=0}^{\pi}\frac{(2\cos\theta\hat{\boldsymbol{r}}+\sin\theta\hat{\boldsymbol{\theta}})\cdot\hat{\boldsymbol{z}}}{r^3}2\pi r^2\sin\theta\,\mathrm{d}\theta\mathrm{d}r$$

$$+\int_{r=d/2}^{\infty}\int_{\theta=\theta_0}^{\pi-\theta_0}\frac{(2\cos\theta\hat{\boldsymbol{r}}+\sin\theta\hat{\boldsymbol{\theta}})\cdot\hat{\boldsymbol{z}}}{r^3}2\pi r^2\sin\theta\,\mathrm{d}\theta\mathrm{d}r\Big\} \quad \leftarrow \cos\theta_0 = d/(2r)$$

$$= -\frac{4\pi R^3 P_o}{9\varepsilon_o} + \frac{2\pi R^3 P_o}{3\varepsilon_o}\Big\{\int_{r=R}^{d/2}\int_{\theta=0}^{\pi}\frac{\sin\theta(2\cos^2\theta-\sin^2\theta)}{r}\mathrm{d}\theta\mathrm{d}r$$

$$+\int_{r=d/2}^{\infty}\int_{\theta=\theta_0}^{\pi-\theta_0}\frac{\sin\theta(2\cos^2\theta-\sin^2\theta)}{r}\mathrm{d}\theta\mathrm{d}r\Big\}$$

$$= -\frac{4\pi R^3 P_o}{9\varepsilon_o} + \frac{2\pi R^3 P_o}{3\varepsilon_o}\Big\{\int_{r=R}^{d/2}\frac{\overbrace{(\cos\theta-\cos^3\theta)\big|_0^{\pi}}^{=0}}{r}\mathrm{d}r + \int_{r=d/2}^{\infty}\frac{(\cos\theta-\cos^3\theta)\big|_{\theta_0}^{\pi-\theta_0}}{r}\mathrm{d}r\Big\}$$

$$= -\frac{4\pi R^3 P_o}{9\varepsilon_o} - \frac{4\pi R^3 P_o}{3\varepsilon_o}\int_{r=d/2}^{\infty}\frac{\cos\theta_0(1-\cos^2\theta_0)}{r}\mathrm{d}r$$

$$= -\frac{4\pi R^3 P_o}{9\varepsilon_o} - \frac{4\pi R^3 P_o}{3\varepsilon_o}\int_{r=d/2}^{\infty}\frac{(d/2r)\,[1-(d/2r)^2]}{r}\mathrm{d}r$$

$$= -\frac{4\pi R^3 P_o}{9\varepsilon_o} - \frac{8\pi R^3 P_o}{9\varepsilon_o} = -\frac{4\pi R^3 P_o}{3\varepsilon_o}.$$

The total electromagnetic momentum is thus seen to be $(4\pi R^3/3)\mu_o P_o J_{so}\hat{\boldsymbol{x}}$. This is equal in magnitude but opposite in direction to the integrated force on the dipole as it grows from zero to the final value of $P_o\hat{\boldsymbol{z}}$, namely,

$$\text{Mechanical momentum} = (4\pi R^3/3)\int_{\text{time}}(\partial\boldsymbol{P}/\partial t)\times\mu_o\boldsymbol{H}\,\mathrm{d}t = (4\pi R^3/3)\mu_o P_o\hat{\boldsymbol{z}}\times J_{so}\hat{\boldsymbol{y}} = -(4\pi R^3/3)\mu_o P_o J_{so}\hat{\boldsymbol{x}}.$$

b) The field momentum in this case is found, once again, by integrating the z-component of the E-field throughout the gap, then cross-multiplying into $(J_{so}/c^2)\hat{\boldsymbol{x}}$. (Due to symmetry, the y-component of the E-field integrates to zero and need not be considered.) The integral of E_z taken at fixed (x,y) along z from $-\infty$ to $+\infty$ now turns out to be exactly zero; see Problem 4-36, where a similar integral for the H-field of a magnetic dipole is evaluated. The total EM momentum within the gap thus vanishes, and we are left to explain the mechanical momentum acquired by the dipole during the (slow) process of spontaneous polarization, namely,

$$\text{Mechanical momentum} = (4\pi R^3/3)\int_{\text{time}} (\partial \boldsymbol{P}/\partial t) \times \mu_o \boldsymbol{H} \mathrm{d}t = (4\pi R^3/3)\mu_o P_o \hat{\boldsymbol{z}} \times J_{so}\hat{\boldsymbol{x}} = (4\pi R^3/3)\mu_o P_o J_{so}\hat{\boldsymbol{y}}.$$

As it turns out, an equal but opposite mechanical momentum is picked up by the parallel plates during the process of spontaneous polarization of the dipole. The Lorentz force of the magnetic field induced by the dipole's time-varying E-field acts on the current sheets to push them both along the negative y-axis. To find the induced H-field, we use Maxwell's 2nd equation, $\boldsymbol{\nabla} \times \boldsymbol{H} = \varepsilon_o \partial \boldsymbol{E}/\partial t$, over a spherical cap at radius $r > R$, extending from $\theta = 0$ to an arbitrary polar angle $\theta = \theta_0$. We will have

$$\int_{\theta=0}^{\theta_0} 2\pi r^2 \sin\theta\, \varepsilon_o E_r(r,\theta)\,\mathrm{d}\theta = \frac{4\pi R^3 P_o}{3r}\int_{\theta=0}^{\theta_0} \sin\theta\cos\theta\,\mathrm{d}\theta = \frac{2\pi R^3 P_o \sin^2\theta_0}{3r}.$$

The time-rate-of-change of the above D-field flux is then equal to the integral of the H-field around the cap's boundary, namely, $2\pi r \sin\theta_0 H_\phi(r,\theta_0,t)$. Denoting by $P_o'(t)$ the time-derivative of $P_o(t)$, we write

$$H_\phi(r,\theta,t) = \frac{R^3 P_o'(t)\sin\theta}{3r^2}.$$

The push (or pull) force on the current sheets is exerted by the x-component of the induced H-field at the surface of each sheet located at $y = \pm d/2$, namely,

$$H_x(r,\theta,\phi,t) = H_\phi(r,\theta,t)\hat{\boldsymbol{\phi}}\cdot\hat{\boldsymbol{x}} = -\frac{R^3 P_o'(t)\sin\theta\sin\phi}{3r^2} = -\frac{R^3 P_o'(t)\, y}{3(x^2+y^2+z^2)^{3/2}}.$$

Integrating the above H_x over an entire xz-plane yields

change of variable: $\rho = y\tan\chi$

$$\iint_{-\infty}^{\infty} H_x(x,y,z,t)\,\mathrm{d}x\,\mathrm{d}z = -\tfrac{1}{3}R^3 P_o'(t)\, y \iint_{-\infty}^{\infty} (x^2+y^2+z^2)^{-3/2}\mathrm{d}x\,\mathrm{d}z$$

$$= -\tfrac{1}{3}R^3 P_o'(t)\, y\int_0^\infty 2\pi\rho(y^2+\rho^2)^{-3/2}\mathrm{d}\rho = -(2\pi R^3/3)P_o'(t)\,\mathrm{sign}(y)\int_0^{\pi/2}\tan\chi(1+\tan^2\chi)^{-1/2}\mathrm{d}\chi$$

$$= -\mathrm{sign}(y)(2\pi R^3/3)P_o'(t)\int_0^{\pi/2}\sin\chi\,\mathrm{d}\chi = -\mathrm{sign}(y)(2\pi R^3/3)P_o'(t).$$

Cross-multiplying the surface-current-density $\pm J_{so}\hat{\boldsymbol{z}}$ into the integrated $\mu_o H_x \hat{\boldsymbol{x}}$ now yields the total force on each sheet as $-(2\pi R^3/3)\mu_o P_o'(t) J_{so}\hat{\boldsymbol{y}}$. This force should be doubled to account for both sheets, then integrated over time to yield the mechanical momentum transferred to the sheets. It is seen that the mechanical momentum is equal in magnitude and opposite in direction to that acquired by the dipole while being polarized.

c) There are many similarities between this part and part (b). The field momentum is found by integrating the z-component of the H-field throughout the gap, then cross-multiplying into $\boldsymbol{E}/c^2 = \mu_0\sigma_{so}\hat{\boldsymbol{y}}$. (Due to symmetry, the x-component of the H-field integrates to zero and need not be considered.) As in Problem 4-36, the integral of H_z taken at fixed (x,y) along z from $-\infty$ to $+\infty$ turns out to be exactly zero. The total EM momentum within the gap thus vanishes, and we are left to explain the mechanical momentum acquired by the dipole during the (slow) process of spontaneous magnetization, namely,

$$\text{Mechanical momentum} = (4\pi R^3/3)\int_{\text{time}} -(\partial\boldsymbol{M}/\partial t)\times\varepsilon_0\boldsymbol{E}\,\mathrm{d}t = -(4\pi R^3/3)\,M_0\hat{\boldsymbol{z}}\times\sigma_{so}\hat{\boldsymbol{y}} = (4\pi R^3/3)M_0\sigma_{so}\hat{\boldsymbol{x}}.$$

It turns out that an equal but opposite mechanical momentum is picked up by the parallel plates during the process of spontaneous magnetization of the dipole. The Lorentz force of the E-field induced by the dipole's time-varying H-field acts on the charged sheets to push them both along the negative x-axis. To find the induced E-field, we use Maxwell's 3rd equation, $\boldsymbol{\nabla}\times\boldsymbol{E} = -\mu_0\partial\boldsymbol{H}/\partial t$, over a spherical cap at radius $r > R$, extending from $\theta = 0$ to an arbitrary polar angle $\theta = \theta_0$. We will have

$$\int_{\theta=0}^{\theta_0} 2\pi r^2\sin\theta\,\mu_0 H_r(r,\theta)\,\mathrm{d}\theta = \frac{4\pi R^3 M_0}{3r}\int_{\theta=0}^{\theta_0}\sin\theta\cos\theta\,\mathrm{d}\theta = \frac{2\pi R^3 M_0\sin^2\theta_0}{3r}.$$

The time-rate-of-change of the above B-field flux is then equal to the integral of the E-field around the cap's boundary, namely, $-2\pi r\sin\theta_0 E_\phi(r,\theta_0,t)$. Denoting by $M_0'(t)$ the time-derivative of $M_0(t)$, we write

$$E_\phi(r,\theta,t) = -\frac{R^3 M_0'(t)\sin\theta}{3r^2}.$$

The push (or pull) force on the current sheets is exerted by the x-component of the induced E-field at the surface of each sheet located at $y = \pm d/2$, namely,

$$E_x(r,\theta,\phi,t) = E_\phi(r,\theta,t)\hat{\boldsymbol{\phi}}\cdot\hat{\boldsymbol{x}} = \frac{R^3 M_0'(t)\sin\theta\sin\phi}{3r^2} = \frac{R^3 M_0'(t)\,y}{3(x^2+y^2+z^2)^{3/2}}.$$

Integrating the above E_x over an entire xz-plane yields

change of variable: $\rho = y\tan\chi$

$$\iint_{-\infty}^{\infty} E_x(x,y,z,t)\,\mathrm{d}x\,\mathrm{d}z = \tfrac{1}{3}R^3 M_0'(t)\,y\iint_{-\infty}^{\infty}(x^2+y^2+z^2)^{-3/2}\mathrm{d}x\,\mathrm{d}z$$
$$= \tfrac{1}{3}R^3 M_0'(t)\,y\int_0^{\infty}2\pi\rho(y^2+\rho^2)^{-3/2}\mathrm{d}\rho = (2\pi R^3/3)\,M_0'(t)\,\mathrm{sign}(y)\int_0^{\pi/2}\tan\chi(1+\tan^2\chi)^{-1/2}\mathrm{d}\chi$$
$$= \mathrm{sign}(y)(2\pi R^3/3)M_0'(t)\int_0^{\pi/2}\sin\chi\,\mathrm{d}\chi = \mathrm{sign}(y)(2\pi R^3/3)M_0'(t).$$

Multiplying the surface-charge-density $\pm\sigma_{so}$ into the integrated $E_x\hat{\boldsymbol{x}}$ now yields the total force on each sheet as $-(2\pi R^3/3)M_0'(t)\sigma_{so}\hat{\boldsymbol{x}}$. This force should be doubled to account for both sheets, then integrated over time to yield the mechanical momentum transferred to the sheets. It is seen that the mechanical momentum is equal in magnitude and opposite in direction to that acquired by the dipole while being magnetized.

d) The sheets in this case pick up exactly the same momentum as in part (c), but the dipole itself will acquire no momentum at all, simply because the E-field does not exert a force on the current-carrying hollow shell. The field momentum, however, is no longer zero because the H-field inside the shell is larger than that in part (c). Considering that the B-field inside the hollow shell must be the same as that inside the solid sphere of part (c), namely,

$$\boldsymbol{B} = \mu_o \boldsymbol{H} + \boldsymbol{M} = -\tfrac{1}{3} M_o \hat{\boldsymbol{z}} + M_o \hat{\boldsymbol{z}} = \tfrac{2}{3} M_o \hat{\boldsymbol{z}},$$

we see that the H-field inside the hollow shell is greater than that inside the solid sphere by $M_o \hat{\boldsymbol{z}}/\mu_o$. The corresponding increase in the field momentum will thus be

$$(4\pi R^3/3)\boldsymbol{E} \times \boldsymbol{H}/c^2 = (4\pi R^3/3)(\sigma_{so}\hat{\boldsymbol{y}}/\varepsilon_o) \times (M_o \hat{\boldsymbol{z}}/\mu_o)\,\varepsilon_o \mu_o = (4\pi R^3/3) M_o \sigma_{so} \hat{\boldsymbol{x}}.$$

This field momentum is equal in magnitude and opposite in direction to that acquired by the parallel plates while the current around the spherical shell rose from zero to its final value. Conservation of momentum is thus confirmed.
