Problem 4-37) a) The two H-fields overlap within the solenoid, where the total energy-density is given by $\frac{1}{2}\mu_o|H_o\hat{\boldsymbol{z}}+\boldsymbol{H}(\boldsymbol{r})|^2$; here $\boldsymbol{H}(\boldsymbol{r})$ is the field produced by the magnetized sphere. The cross-term is thus $\mu_o H_o H_z(\boldsymbol{r})$, and to show that it does *not* contribute to the total H-field energy, all we need to demonstrate is that $\int_0^\infty H_z(\rho=\rho_o,\phi=\phi_o,z)\,\mathrm{d}z=0$, where H_z is now expressed in the cylindrical (ρ,ϕ,z) coordinates. The z-component of the magnetized sphere's H-field may be written as follows:

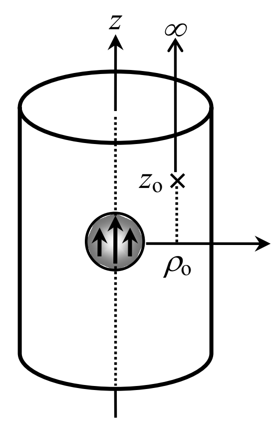
$$H_z(\boldsymbol{r})=\begin{cases}\dfrac{M_oR^3(2\cos\theta\,\hat{\boldsymbol{r}}+\sin\theta\,\hat{\boldsymbol{\theta}})\cdot\hat{\boldsymbol{z}}}{3\mu_o r^3}=\dfrac{M_oR^3(2\cos^2\theta-\sin^2\theta)}{3\mu_o r^3}=\dfrac{M_oR^3(2z^2-\rho_o^2)}{3\mu_o(\rho_o^2+z^2)^{5/2}}; & r>R,\\ -\dfrac{M_o}{3\mu_o}; & r<R.\end{cases}$$

The desired integral along the z-axis, evaluated from an arbitrary point $z=z_o$ (to be specified later) to $z=\infty$, is now obtained by a change of variable, $z=\rho_o\tan\chi$, as follows:

$$\int_{z_o}^{\infty}H_z(\rho_o,\phi_o,z)\,\mathrm{d}z=\frac{M_oR^3}{3\mu_o}\int_{z_o}^{\infty}\frac{(2z^2-\rho_o^2)}{(\rho_o^2+z^2)^{5/2}}\mathrm{d}z=\frac{M_oR^3}{3\mu_o\rho_o^2}\int_{\chi_o}^{\pi/2}\frac{(2\tan^2\chi-1)}{(1+\tan^2\chi)^{3/2}}\mathrm{d}\chi$$

$$=\frac{M_oR^3}{3\mu_o\rho_o^2}\int_{\chi_o}^{\pi/2}\cos\chi\,(3\sin^2\chi-1)\,\mathrm{d}\chi=\frac{M_oR^3}{3\mu_o\rho_o^2}(\sin^3\chi-\sin\chi)\Big|_{\chi_o}^{\pi/2}=\frac{M_oR^3\sin\chi_o\cos^2\chi_o}{3\mu_o\rho_o^2}.$$

If $\rho_o\ge R$, the lower limit of the above integral will be $z_o=0$, corresponding to $\chi_o=0$, in which case the integral vanishes. For $\rho_o<R$, we have $z_o=\sqrt{R^2-\rho_o^2}$, i.e., on the surface of the sphere, for which $\sin\chi_o=z_o/R$ and $\cos\chi_o=\rho_o/R$. The evaluated integral, $M_oz_o/(3\mu_o)$, will then be cancelled out by the integral of the uniform H_z inside the sphere, from $z=0$ to z_o. Therefore, the contribution of the cross-term, $\iiint_{\substack{\text{solenoid}\\ \text{volume}}}\mu_oH_oH_z(\boldsymbol{r})\mathrm{d}\boldsymbol{r}$, to the overall H-field energy will be precisely zero.



The point of this exercise is to demonstrate that all the electrical energy spent in bringing up the solenoid's current from zero to its final value, goes into building the magnetic field inside the solenoid; no energy is given to (or taken away from) the magnetized sphere, so long as it maintains its magnetization.

b) Inside the small sphere, the H-fields are $-M_o\hat{\boldsymbol{z}}/(3\mu_o)$ and $2M_s\hat{\boldsymbol{z}}/(3\mu_o)$. Their dot-product, integrated over the volume of the small sphere, is thus given by $-8\pi R^3M_oM_s/(27\mu_o^2)$.

In the region between the two spheres, The H-field of the larger sphere is uniform, while that of the smaller sphere varies with r and θ. The integrated cross-term is evaluated as follows:

$$\int_{r=R}^{R_s}\int_{\theta=0}^{\pi}\frac{2M_s\hat{\boldsymbol{z}}}{3\mu_o}\cdot\frac{M_oR^3(2\cos\theta\,\hat{\boldsymbol{r}}+\sin\theta\,\hat{\boldsymbol{\theta}})}{3\mu_o r^3}2\pi r^2\sin\theta\mathrm{d}r\mathrm{d}\theta$$

$$=\frac{4\pi M_oM_sR^3}{9\mu_o^2}\int_{r=R}^{R_s}r^{-1}\mathrm{d}r\int_{\theta=0}^{\pi}(2\cos^2\theta-\sin^2\theta)\sin\theta\mathrm{d}\theta$$

$$=\frac{4\pi M_oM_sR^3}{9\mu_o^2}\ln(R_s/R)\,(\cos\theta-\cos^3\theta)\Big|_{\theta=0}^{\pi}=0.$$

Outside the large sphere, both H-fields are functions of r and θ, and their integrated cross-term becomes

$$\int_{r=R_s}^{\infty}\int_{\theta=0}^{\pi}\frac{M_oR^3(2\cos\theta\,\hat{\boldsymbol{r}}+\sin\theta\,\hat{\boldsymbol{\theta}})}{3\mu_o r^3}\cdot\frac{M_sR_s^3(2\cos\theta\,\hat{\boldsymbol{r}}+\sin\theta\,\hat{\boldsymbol{\theta}})}{3\mu_o r^3}2\pi r^2\sin\theta\mathrm{d}r\mathrm{d}\theta$$

$$=\frac{2\pi M_oM_sR^3R_s^3}{9\mu_o^2}\int_{r=R_s}^{\infty}r^{-4}\mathrm{d}r\int_{\theta=0}^{\pi}(4\cos^2\theta+\sin^2\theta)\sin\theta\,\mathrm{d}\theta$$

$$=\frac{2\pi M_oM_sR^3R_s^3}{9\mu_o^2}(3R_s^3)^{-1}(-\cos^3\theta-\cos\theta)\Big|_{\theta=0}^{\pi}=\frac{8\pi R^3M_oM_s}{27\mu_o^2}.$$

The three contributions to the integrated cross-term are thus seen to add up to zero, confirming that the total H-field energy of the two spheres is the sum of their individual energies.
