Problem 4-36)

a) $$\int_{-\infty}^{\infty}\mathcal{E}(\boldsymbol{r})\mathrm{d}\boldsymbol{r}=\tfrac{1}{2}\mu_o\int_{-\infty}^{\infty}|\boldsymbol{H}|^2\mathrm{d}\boldsymbol{r}=\underbrace{\frac{M_o^2(4\pi R^3/3)}{18\mu_o}}_{\text{Inside the sphere}}+\underbrace{\frac{M_o^2R^6}{18\mu_o}\int_{r=R}^{\infty}\int_{\theta=0}^{\pi}\frac{4\cos^2\theta+\sin^2\theta}{r^6}2\pi r^2\sin\theta\,\mathrm{d}\theta\mathrm{d}\boldsymbol{r}}_{\text{Outside the sphere}}$$

$$=\frac{M_o^2(4\pi R^3/3)}{18\mu_o}+\frac{2\pi M_o^2R^6}{18\mu_o}\int_{r=R}^{\infty}\int_{\theta=0}^{\pi}\frac{(3\cos^2\theta+1)\sin\theta}{r^4}\,\mathrm{d}\theta\mathrm{d}\boldsymbol{r}$$

$$=\frac{M_o^2(4\pi R^3/3)}{18\mu_o}+\frac{2\pi M_o^2R^6}{18\mu_o}\left(-\frac{1}{3r^3}\right)\Big|_{r=R}^{\infty}(-\cos^3\theta-\cos\theta)\Big|_0^{\pi}$$

$$=\frac{M_o^2(4\pi R^3/3)}{18\mu_o}+\frac{2M_o^2(4\pi R^3/3)}{18\mu_o}$$

$$=\frac{M_o^2(4\pi R^3/3)}{6\mu_o}.$$

The H-field energy is thus seen to be divided between the inside and outside regions of the magnetized sphere, with the inside region containing one-third of the total energy. Denoting the dipole moment by $\boldsymbol{m}=(4\pi R^3/3)M_o\hat{\boldsymbol{z}}$, the total energy may also be expressed as $m^2/(8\pi\mu_o R^3)$. This so-called self-energy of the dipole increases without bound as the sphere radius shrinks while $\boldsymbol{m}$ is kept constant.

b) The time rate of exchange of electromagnetic energy density between a magnetic material having magnetization $\boldsymbol{M}(\boldsymbol{r},t)$ and a magnetic field $\boldsymbol{H}(\boldsymbol{r},t)$ is given by

$$\frac{\partial\mathcal{E}(\boldsymbol{r},t)}{\partial t}=\boldsymbol{H}(\boldsymbol{r},t)\cdot\frac{\partial\boldsymbol{M}(\boldsymbol{r},t)}{\partial t}.$$

If the magnetization throughout the sphere rises uniformly and slowly, the H-field acting on this magnetization will also be uniform and given by $\boldsymbol{H}(t)=-\boldsymbol{M}(t)/3\mu_o$. If the magnetization changes by a small amount $\Delta\boldsymbol{M}$ during a time interval Δt, the exchanged energy density will be given by $\Delta\mathcal{E}=\boldsymbol{H}\cdot\Delta\boldsymbol{M}$, independent of Δt, provided of course that $\Delta\boldsymbol{M}/\Delta t$ is sufficiently small for the effects of radiation to be negligible. Thus, when $\boldsymbol{M}$ rises from an initial value of zero to a final value of $M_o\hat{\boldsymbol{z}}$, the total exchanged energy will be

$$(4\pi R^3/3)\int_{\boldsymbol{M}=0}^{M_o\hat{\boldsymbol{z}}}\boldsymbol{H}\cdot\mathrm{d}\boldsymbol{M}=-(4\pi R^3/3)\int_0^{M_o}(M/3\mu_o)\mathrm{d}M=-\frac{M_o^2(4\pi R^3/3)}{6\mu_o}.$$

The minus sign in the above expression (caused by the internal H-field being opposite in direction to $\boldsymbol{M}$), indicates that the energy has come out of the magnetized material and appeared as H-field energy both inside and outside the sphere. Aside from this minus sign, the final expression is identical to that obtained in part (a) by integrating the H-field energy density of the spherical dipole over the entire space.

c) We determine the bound electric current-density of the solid sphere by expressing its magnetization as $\boldsymbol{M}(\boldsymbol{r},t)=M_{\text{o}}\hat{\boldsymbol{z}}\,\text{Sphere}(r/R)$, which in spherical coordinates could be written as $\boldsymbol{M}(\boldsymbol{r},t)=M_{\text{o}}\text{Sphere}(r/R)(\cos\theta\hat{\boldsymbol{r}}-\sin\theta\hat{\boldsymbol{\theta}})$. We will have

$$\boldsymbol{J}_{\text{bound}}^{(e)}(\boldsymbol{r},t)=\mu_{\text{o}}^{-1}\boldsymbol{\nabla}\times\boldsymbol{M}(\boldsymbol{r},t)=\frac{\mu_{\text{o}}^{-1}}{r}\left[\frac{\partial(r\,M_\theta)}{\partial r}-\frac{\partial M_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}}$$

$$=\frac{\mu_{\text{o}}^{-1}M_{\text{o}}}{r}\{-[\text{Sphere}(r/R)-r\delta(r-R)]\sin\theta+\text{Sphere}(r/R)\sin\theta\}\hat{\boldsymbol{\phi}}$$

$$=\mu_{\text{o}}^{-1}M_{\text{o}}\delta(r-R)\sin\theta\hat{\boldsymbol{\phi}}.$$

This azimuthal current-density confined to the surface of the sphere may equivalently be described as a surface-current-density $\boldsymbol{J}_s(r=R,\theta,\phi)=\mu_{\text{o}}^{-1}M_{\text{o}}\sin\theta\,\hat{\boldsymbol{\phi}}$.

Now, the magnetic field distribution of a uniformly-magnetized sphere may be obtained by solving Maxwell's 2nd and 4th equations in two different (albeit equivalent) ways: (i) by writing them as $\boldsymbol{\nabla}\times\boldsymbol{H}(\boldsymbol{r})=0$ and $\mu_{\text{o}}\boldsymbol{\nabla}\cdot\boldsymbol{H}(\boldsymbol{r})=-\boldsymbol{\nabla}\cdot\boldsymbol{M}(\boldsymbol{r})=\rho_{\text{bound}}^{(m)}$, and (ii) by writing them as $\boldsymbol{\nabla}\times\boldsymbol{B}(\boldsymbol{r})=\boldsymbol{\nabla}\times\boldsymbol{M}(\boldsymbol{r})=\mu_{\text{o}}\boldsymbol{J}_{\text{bound}}^{(e)}(\boldsymbol{r})$ and $\boldsymbol{\nabla}\cdot\boldsymbol{B}(\boldsymbol{r})=0$. In both cases we will find the same solution for $\boldsymbol{H}(\boldsymbol{r})$ as given in the statement of the problem. The B-field inside the sphere will thus turn out to be $\boldsymbol{B}(\boldsymbol{r})=\mu_{\text{o}}\boldsymbol{H}(\boldsymbol{r})+\boldsymbol{M}(\boldsymbol{r})=\tfrac{2}{3}\boldsymbol{M}(\boldsymbol{r})=\tfrac{2}{3}M_{\text{o}}\hat{\boldsymbol{z}}$.

In the case of a hollow spherical shell carrying a constant surface-current-density $\boldsymbol{J}_s(r=R,\theta,\phi)$, the relevant Maxwell equations are $\boldsymbol{\nabla}\times\boldsymbol{B}(\boldsymbol{r})=\mu_{\text{o}}\boldsymbol{J}_s(\theta,\phi)\delta(r-R)$ and $\boldsymbol{\nabla}\cdot\boldsymbol{B}(\boldsymbol{r})=0$, which are the same as the aforementioned second set of equations for a uniformly-magnetized sphere. The B-field inside as well as outside the hollow shell must therefore be identical with the B-field of the uniformly-magnetized sphere. We conclude that, while the H-field outside the hollow shell is the same as that of the uniformly-magnetized solid sphere, the H-field inside the hollow shell is given by $\boldsymbol{H}(\boldsymbol{r})=\mu_{\text{o}}^{-1}\boldsymbol{B}(\boldsymbol{r})=2\boldsymbol{M}(\boldsymbol{r})/(3\mu_{\text{o}})$. The H-field energy density inside the hollow shell is thus four times greater than that inside the uniformly-magnetized solid sphere. When the procedure used in part (a) is repeated for the hollow shell, the resulting total energy of the H-field is found to be $(4\pi R^3/3)M_{\text{o}}^2/(3\mu_{\text{o}})$, which is twice as large as that of the solid sphere.

d) Using Maxwell's 3rd equation, $\boldsymbol{\nabla}\times\boldsymbol{E}(\boldsymbol{r},t)=-\partial\boldsymbol{B}(\boldsymbol{r},t)/\partial t$, and the fact that the B-field inside the hollow shell is uniform and equal to $\tfrac{2}{3}M_{\text{o}}\hat{\boldsymbol{z}}$, we find the induced E-field on the ring of radius $R\sin\theta$ to be given by

$$2\pi R\sin\theta E_\phi(r=R,\theta,\phi)=-(\pi R^2\sin^2\theta)\frac{\text{d}}{\text{d}t}[\tfrac{2}{3}M(t)]\quad\rightarrow\quad E_\phi(r=R,\theta,\phi)=-\tfrac{1}{3}R\sin\theta\frac{\text{d}M(t)}{\text{d}t}.$$

The rate of exchange of EM energy density with the surface-current $\boldsymbol{J}_s(r=R,\theta,\phi)$ is thus found to be

$$\boldsymbol{E}(r=R,\theta,\phi)\cdot\boldsymbol{J}_s(r=R,\theta,\phi)=-\tfrac{1}{3}R\sin\theta\frac{\text{d}M(t)}{\text{d}t}\mu_{\text{o}}^{-1}M(t)\sin\theta\;=-\frac{R\sin^2\theta}{6\mu_{\text{o}}}\frac{\text{d}M^2(t)}{\text{d}t}.$$

Integration over the surface of the sphere then yields

$$\int_{\theta=0}^{\pi} 2\pi R^2 \sin\theta\, \boldsymbol{E}(r=R,\theta,\phi)\cdot \boldsymbol{J}_s(r=R,\theta,\phi)\,\mathrm{d}\theta = -\frac{\pi R^3}{3\mu_o}\frac{\mathrm{d}M^2(t)}{\mathrm{d}t}\int_{\theta=0}^{\pi}\sin^3\theta\,\mathrm{d}\theta = -\frac{(4\pi R^3/3)}{3\mu_o}\frac{\mathrm{d}M^2(t)}{\mathrm{d}t}.$$

Integrating the above result from $M=0$ to M_o yields a total energy of $(4\pi R^3/3)M_o^2/(3\mu_o)$, which is the energy extracted from the surface-current while this current is being raised from zero to its final value. The total energy extracted from the surface current of course goes into building the H-field both inside and outside the shell, as shown in part (c).
