Problem 34) The polarization of the point-dipole is $\boldsymbol{P}(\boldsymbol{r},t) = p_o \hat{\boldsymbol{z}} \delta[x - x_o(t)]\delta(y)\delta(z)$, where $x_o(t)$ is the location of the dipole on the x-axis at time t. Initially, $x_o(t)$ is $-\infty$, but as time goes on, the gradient force of the E-field pulls the dipole closer to the capacitor.

a) In the absence of magnetic fields, the EM force-density is given by $\boldsymbol{F}(\boldsymbol{r},t) = [\boldsymbol{P}(\boldsymbol{r},t)\cdot\boldsymbol{\nabla}]\boldsymbol{E}(\boldsymbol{r},t)$. Considering that the dipole moves slowly, we ignore the effect of its own radiation and write

$$\boldsymbol{F}(\boldsymbol{r},t) = p_o \delta[x - x_o(t)]\delta(y)\delta(z)\partial\boldsymbol{E}(\boldsymbol{r})/\partial z. \qquad (1)$$

← The capacitor's E-field is time-independent.

The total force experienced by the dipole at time t is thus obtained by integrating Eq.(1) over the entire space, namely,

$$\boldsymbol{f}(t) = \int_{-\infty}^{\infty} \boldsymbol{F}(\boldsymbol{r},t)\,\mathrm{d}\boldsymbol{r} = p_o \partial\boldsymbol{E}(\boldsymbol{r})/\partial z\big|_{x=x_o(t),\, y=0,\, z=0} = p_o(\partial/\partial z)E_x[x = x_o(t), y = 0, z = 0]\hat{\boldsymbol{x}}. \qquad (2)$$

↑ At the location of the dipole, E_y and E_z do not vary with z.

For the static field of the capacitor, $\boldsymbol{\nabla}\times\boldsymbol{E} = 0$ yields $\partial E_x/\partial z = \partial E_z/\partial x$. Therefore,

$$\boldsymbol{f}(t) = p_o(\partial/\partial x)E_z[x = x_o(t), y = 0, z = 0]\hat{\boldsymbol{x}}. \qquad (3)$$

The force on the dipole is thus seen to be a function of its position; as such, it may be written $f_x(x_o) = p_o(\partial/\partial x)E_z(x_o, 0, 0)$. Since the work done on the dipole when it moves by a distance Δx along the x-axis is given by $f_x(x_o)\Delta x$, the work done in moving from $x = -\infty$ to $x = 0$ is given by

$$\int_{-\infty}^{0} f_x(x_o)\,\mathrm{d}x_o = p_o \int_{-\infty}^{0} [\partial E_z(x, 0, 0)/\partial x]\,\mathrm{d}x = p_o E_z(0,0,0) = p_o E_o. \qquad (4)$$

b) Denoting the energy-density exchanged between the field and the dipole by $\mathcal{E}(\boldsymbol{r},t)$, we have

$$\partial\mathcal{E}(\boldsymbol{r},t)/\partial t = \boldsymbol{E}(\boldsymbol{r},t)\cdot\partial\boldsymbol{P}(\boldsymbol{r},t)/\partial t = -p_o E_z(\boldsymbol{r})\, x_o'(t)\delta'[x - x_o(t)]\delta(y)\delta(z). \qquad (5)$$

↑ Derivative of $x_o(t)$ with respect to time.

↑ Derivative of $\delta(x)$ with respect to x.

Integrating the above equation over the entire space at fixed time t, and denoting by $\underset{\sim}{\mathcal{E}}(t)$ the integrated $\mathcal{E}(\boldsymbol{r},t)$, we find the time-rate-of-exchange of energy between the E-field and the dipole as follows:

$$\partial\underset{\sim}{\mathcal{E}}(t)/\partial t = p_o x_o'(t)(\partial/\partial x)E_z[x = x_o(t), y = 0, z = 0] = p_o(\partial/\partial t)E_z[x_o(t), 0, 0]. \qquad (6)$$

Integrating over time now yields the energy gained by the dipole in going from $x = -\infty$ to $x = 0$ at center of the capacitor as $p_o E_o$.

c) In Problem 23 it was shown that the energy content of an ideal capacitor whose E-field is $E_o\hat{\boldsymbol{z}}$ would be reduced by $p_o E_o$ if a spherical dipole $p_o\hat{\boldsymbol{z}}$ were placed anywhere inside the capacitor. This is because the E-field of the dipole inside the sphere adds to the capacitor's field, while, on average, the dipolar E-field outside the sphere subtracts from that of the capacitor. When the net

E-field is put into the expression $\frac{1}{2}\varepsilon_o|\boldsymbol{E}|^2$ of the energy-density and integrated over all space, the cross-term between the two fields causes a reduction of the total energy by p_oE_o. The kinetic energy gained by the dipole is thus accounted for by the reduced overall energy of the E-field.
