Solutions **Opti 501**

Problem 32)

a) Denoting by "a" the cross-sectional area of the wire, we find $\vec{J}(z,t) = (I_0/a)\sin(\omega t - \kappa z)\hat{z}$. The continuity equation $\vec{\nabla}\cdot\vec{J} + \frac{\partial \rho}{\partial t} = 0$ thus yields: $\frac{\partial \rho(z,t)}{\partial t} = -\frac{\partial J_z(z,t)}{\partial z} = (\kappa I_0/a)\cos(\omega t - \kappa z) \Rightarrow$

$\rho(z,t) = \frac{\kappa I_0}{\omega a}\sin(\omega t - \kappa z) \Rightarrow$ linear charge density $\lambda(z,t) = a\rho(z,t) = \frac{\kappa I_0}{\omega}\sin(\omega t - \kappa z)$.

b) $$\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial \psi}{\partial t} = \frac{\partial A_z}{\partial z} + \mu_0\epsilon_0\frac{\partial \psi}{\partial t} = -\frac{\mu_0 I_0}{4}\left\{-\kappa Y_0(\sqrt{k_0^2-\kappa^2}\rho)\cos(\omega t-\kappa z) + \kappa J_0(\sqrt{k_0^2-\kappa^2}\rho)\sin(\omega t-\kappa z)\right\} - \frac{\mu_0 \kappa I_0}{4\omega}\left\{\omega Y_0(\sqrt{k_0^2-\kappa^2}\rho)\cos(\omega t-\kappa z) - \omega J_0(\sqrt{k_0^2-\kappa^2}\rho)\sin(\omega t-\kappa z)\right\} = 0$$ ✓ ← Lorentz Gauge

c) $$\vec{E}(\rho,z,t) = -\vec{\nabla}\psi - \frac{\partial \vec{A}}{\partial t} = -\frac{\partial \psi}{\partial \rho}\hat{\rho} - \left(\frac{\partial \psi}{\partial z} + \frac{\partial A_z}{\partial t}\right)\hat{z} \Rightarrow$$

$$E_\rho(\rho,z,t) = -\frac{\kappa I_0\sqrt{k_0^2-\kappa^2}}{4\epsilon_0\omega}\left\{Y_1(\sqrt{k_0^2-\kappa^2}\rho)\sin(\omega t-\kappa z) + J_1(\sqrt{k_0^2-\kappa^2}\rho)\cos(\omega t-\kappa z)\right\}$$ ✓

$$E_z(\rho,z,t) = \left(-\frac{\kappa^2 I_0}{4\epsilon_0\omega} + \frac{\mu_0 I_0\omega}{4}\right)\left\{Y_0(\sqrt{k_0^2-\kappa^2}\rho)\cos(\omega t-\kappa z) - J_0(\sqrt{k_0^2-\kappa^2}\rho)\sin(\omega t-\kappa z)\right\}$$

$$= \frac{(k_0^2-\kappa^2)I_0}{4\epsilon_0\omega}\left\{Y_0(\sqrt{k_0^2-\kappa^2}\rho)\cos(\omega t-\kappa z) - J_0(\sqrt{k_0^2-\kappa^2}\rho)\sin(\omega t-\kappa z)\right\}$$ ✓

$$\vec{H}(\rho,z,t) = \frac{1}{\mu_0}\vec{\nabla}\times\vec{A} = -\frac{1}{\mu_0}\frac{\partial A_z}{\partial \rho}\hat{\phi} \Rightarrow$$

$$H_\phi(\rho,z,t) = -\frac{1}{4}I_0\sqrt{k_0^2-\kappa^2}\left\{Y_1(\sqrt{k_0^2-\kappa^2}\rho)\sin(\omega t-\kappa z) + J_1(\sqrt{k_0^2-\kappa^2}\rho)\cos(\omega t-\kappa z)\right\}$$ ✓

d) In the limit of large ρ we'll have: $J_n(x) \to \sqrt{\frac{2}{\pi x}} \cos(x - \frac{n\pi}{2} - \frac{\pi}{4})$ and $Y_n(x) \to \sqrt{\frac{2}{\pi x}} \sin(x - \frac{n\pi}{2} - \frac{\pi}{4})$. Therefore,

$$E_\rho(\rho,z,t) \to -\frac{\kappa I_0 \sqrt{k_0^2-\kappa^2}}{4\epsilon_0 \omega} \sqrt{\frac{2}{\pi\sqrt{k_0^2-\kappa^2}\,\rho}} \left\{ \sin\left(\sqrt{k_0^2-\kappa^2}\,\rho - \frac{3\pi}{4}\right)\sin(\omega t - \kappa z) + \cos\left(\sqrt{k_0^2-\kappa^2}\,\rho - \frac{3\pi}{4}\right)\cos(\omega t - \kappa z)\right\} = \frac{\kappa I_0 \sqrt[4]{k_0^2-\kappa^2}}{2\epsilon_0\omega\sqrt{2\pi\rho}} \cos\left(\sqrt{k_0^2-\kappa^2}\,\rho + \kappa z - \omega t + \frac{\pi}{4}\right)$$

$$E_z(\rho,z,t) \to \frac{-I_0 (k_0^2-\kappa^2)^{3/4}}{2\epsilon_0\omega\sqrt{2\pi\rho}} \cos\left(\sqrt{k_0^2-\kappa^2}\,\rho + \kappa z - \omega t + \pi/4\right)$$

$$H_\phi(\rho,z,t) \to \frac{+I_0 \sqrt[4]{k_0^2-\kappa^2}}{2\sqrt{2\pi\rho}} \cos\left(\sqrt{k_0^2-\kappa^2}\,\rho + \kappa z - \omega t + \pi/4\right)$$

$$\vec{S}(\rho,z,t) = (E_\rho\hat{\rho} + E_z\hat{z}) \times H_\phi\hat{\phi} = -E_z H_\phi \hat{\rho} + E_\rho H_\phi \hat{z}$$

For $\rho \to \infty$ we'll have $\vec{S}(\rho,z,t) \to \frac{\sqrt{k_0^2-\kappa^2}\, I_0^2}{8\pi\epsilon_0\omega\rho} \left(\sqrt{k_0^2-\kappa^2}\,\hat{\rho} + \kappa\hat{z}\right)\cos^2\left(\sqrt{k_0^2-\kappa^2}\,\rho + \kappa z - \omega t + \pi/4\right)$