Opti 501 Solutions

31) a) $\vec{B} = \vec{\nabla}\times\vec{A}' = \vec{\nabla}\times(\vec{A}+\vec{\nabla}f) = \vec{\nabla}\times\vec{A} + \overbrace{\vec{\nabla}\times\vec{\nabla}f}^{0} = \vec{\nabla}\times\vec{A}$ ✓

(Because $\vec{\nabla}\times\vec{\nabla}f = 0$ for any function f.)

b) $\vec{E} = -\vec{\nabla}\psi' - \frac{\partial}{\partial t}\vec{A}' = -\vec{\nabla}(\psi - \frac{\partial f}{\partial t}) - \frac{\partial}{\partial t}(\vec{A}+\vec{\nabla}f) = -\vec{\nabla}\psi - \frac{\partial}{\partial t}\vec{A} + \cancel{\frac{\partial}{\partial t}\vec{\nabla}f} - \cancel{\frac{\partial}{\partial t}\vec{\nabla}f}$

$\Rightarrow \vec{E} = -\vec{\nabla}\psi - \frac{\partial}{\partial t}\vec{A}$ ✓

c) $\vec{\nabla}\cdot\vec{A}' + \frac{1}{c^2}\frac{\partial\psi'}{\partial t} = \vec{\nabla}\cdot\vec{A} + \vec{\nabla}\cdot\vec{\nabla}f + \frac{1}{c^2}\frac{\partial\psi}{\partial t} - \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} = \overbrace{\cancel{(\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial\psi}{\partial t})}}^{0} + (\vec{\nabla}^2 f - \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2}) = 0$

$\Rightarrow f(\vec{r},t)$ must be a solution of the second-order differential equation $\vec{\nabla}^2 f = \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2}$. ✓

d) $\vec{\nabla}\cdot\vec{A}' = \vec{\nabla}\cdot\vec{A} + \vec{\nabla}\cdot\vec{\nabla}f = 0 \Rightarrow \vec{\nabla}^2 f = -\vec{\nabla}\cdot\vec{A} = \frac{1}{c^2}\frac{\partial\psi}{\partial t}$. ✓

Note that, in the Lorentz gauge, $\vec{\nabla}^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = -\rho/\epsilon_0 \Rightarrow \vec{\nabla}^2\psi - \frac{\partial}{\partial t}\vec{\nabla}^2 f = -\rho/\epsilon_0$

$\Rightarrow \vec{\nabla}^2(\psi - \frac{\partial f}{\partial t}) = -\rho/\epsilon_0 \Rightarrow \vec{\nabla}^2\psi' = -\rho/\epsilon_0$. Thus, in the Coulomb gauge, the scalar potential $\psi'(\vec{r},t)$ is obtained from the charge density distribution $\rho(\vec{r},t)$ in exactly the same way as in electrostatics. In other words, the delay $|\vec{r}-\vec{r}'|/c$ due to the finite travel time between $\vec{r}$ and $\vec{r}'$ is ignored. As for the vector potential $\vec{A}'(\vec{r},t)$, we find the following differential equation starting with Maxwell's 2nd equation (assuming $\vec{P}=0$, $\vec{M}=0$):

$\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t} \Rightarrow \vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} \Rightarrow \vec{\nabla}\times(\vec{\nabla}\times\vec{A}') = \mu_0\vec{J} + \frac{1}{c^2}\frac{\partial}{\partial t}(-\vec{\nabla}\psi' - \frac{\partial\vec{A}'}{\partial t})$

$\Rightarrow \vec{\nabla}(\underbrace{\cancel{\vec{\nabla}\cdot\vec{A}'}}_{0}) - \vec{\nabla}^2\vec{A}' = \mu_0\vec{J} - \frac{1}{c^2}\frac{\partial}{\partial t}\vec{\nabla}\psi' - \frac{1}{c^2}\frac{\partial^2\vec{A}'}{\partial t^2} \Rightarrow \vec{\nabla}^2\vec{A}' - \frac{1}{c^2}\frac{\partial^2\vec{A}'}{\partial t^2} = -\mu_0\vec{J} + \frac{1}{c^2}\frac{\partial}{\partial t}\vec{\nabla}\psi'$. ✓

The second-order partial differential equation for $\vec{A}'$ thus involves as source terms not only $\vec{J}$ but also $\frac{\partial}{\partial t}\vec{\nabla}\psi'$, which is derived from the charge distribution $\rho(\vec{r},t)$.