Opti 501 **Solutions**

Problem 26)

a) $\rho(\boldsymbol{r},t) = q\,\delta(x-Vt)\,\delta(y)\,\delta(z)$; $\boldsymbol{J}(\boldsymbol{r},t) = qV\,\delta(x-Vt)\,\delta(y)\,\delta(z)\,\hat{\boldsymbol{x}}$.

b) $\rho(\boldsymbol{k},\omega) = \iiiint_{-\infty}^{\infty} \rho(\boldsymbol{r},t)\exp[-\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\,\mathrm{d}\boldsymbol{r}\,\mathrm{d}t = q\iint_{-\infty}^{\infty} \delta(x-Vt)\exp[-\mathrm{i}(k_x x-\omega t)]\,\mathrm{d}x\,\mathrm{d}t$

$= q\int_{-\infty}^{\infty} \exp[\mathrm{i}(\omega - Vk_x)t]\,\mathrm{d}t = 2\pi q\,\delta(\omega - Vk_x)$.

$\boldsymbol{J}(\boldsymbol{k},\omega) = \iiiint_{-\infty}^{\infty} \boldsymbol{J}(\boldsymbol{r},t)\exp[-\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\,\mathrm{d}\boldsymbol{r}\,\mathrm{d}t = qV\hat{\boldsymbol{x}}\iint_{-\infty}^{\infty} \delta(x-Vt)\exp[-\mathrm{i}(k_x x-\omega t)]\,\mathrm{d}x\,\mathrm{d}t$

$= qV\hat{\boldsymbol{x}}\int_{-\infty}^{\infty} \exp[\mathrm{i}(\omega - Vk_x)t]\,\mathrm{d}t = 2\pi qV\,\delta(\omega - Vk_x)\,\hat{\boldsymbol{x}}$.

The scalar and vector potentials are thus given by

$$\psi(\boldsymbol{k},\omega) = \varepsilon_o^{-1}\rho(\boldsymbol{k},\omega)/(k^2-\omega^2/c^2) = (2\pi q/\varepsilon_o)\delta(\omega - Vk_x)/(k^2-\omega^2/c^2);$$
$$\boldsymbol{A}(\boldsymbol{k},\omega) = \mu_o\boldsymbol{J}(\boldsymbol{k},\omega)/(k^2-\omega^2/c^2) = (2\pi\mu_o qV\hat{\boldsymbol{x}})\,\delta(\omega - Vk_x)/(k^2-\omega^2/c^2).$$

c) Inverse Fourier transforming the scalar potential, we find

$\psi(\boldsymbol{r},t) = (2\pi)^{-4}\iiiint_{-\infty}^{\infty} \psi(\boldsymbol{k},\omega)\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\,\mathrm{d}\boldsymbol{k}\,\mathrm{d}\omega$

$= (2\pi)^{-3}(q/\varepsilon_o)\iiiint_{-\infty}^{\infty} (k^2-\omega^2/c^2)^{-1}\delta(\omega - Vk_x)\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\,\mathrm{d}\boldsymbol{k}\,\mathrm{d}\omega$

$= (2\pi)^{-3}(q/\varepsilon_o)\iiint_{-\infty}^{\infty} [(1-V^2/c^2)k_x^2 + k_y^2 + k_z^2]^{-1}\exp\{\mathrm{i}[k_x(x-Vt)+k_y y+k_z z]\}\,\mathrm{d}k_x\,\mathrm{d}k_y\,\mathrm{d}k_z$

Defining the parameter $\gamma = 1/\sqrt{1-(V/c)^2}$, then changing the variable from k_x to k_x/γ yields

$\psi(\boldsymbol{r},t) = (2\pi)^{-3}(\gamma q/\varepsilon_o)\iiint_{-\infty}^{\infty} (k_x^2 + k_y^2 + k_z^2)^{-1}\exp\{\mathrm{i}[k_x\gamma(x-Vt)+k_y y+k_z z]\}\,\mathrm{d}k_x\,\mathrm{d}k_y\,\mathrm{d}k_z$

$= (2\pi)^{-3}(\gamma q/\varepsilon_o)\iiint_{-\infty}^{\infty} k^{-2}\exp\{\mathrm{i}\boldsymbol{k}\cdot[\gamma(x-Vt)\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}]\}\,\mathrm{d}\boldsymbol{k}$

$= (2\pi)^{-2}(\gamma q/\varepsilon_o)\int_0^{\infty}\mathrm{d}k\int_0^{\pi}\sin\theta\,\exp[\mathrm{i}k\sqrt{\gamma^2(x-Vt)^2+y^2+z^2}\,\cos\theta]\,\mathrm{d}\theta$

$= (\gamma q/2\pi^2\varepsilon_o)\int_0^{\infty}\{\sin[k\sqrt{\gamma^2(x-Vt)^2+y^2+z^2}\,]/[k\sqrt{\gamma^2(x-Vt)^2+y^2+z^2}\,]\}\,\mathrm{d}k$

$= \gamma q/[4\pi\varepsilon_o\sqrt{\gamma^2(x-Vt)^2+y^2+z^2}\,]$

Similarly, the inverse Fourier transform of $\boldsymbol{A}(\boldsymbol{k},\omega)$ is found to be

$$\boldsymbol{A}(\boldsymbol{r},t) = \mu_o\gamma qV\hat{\boldsymbol{x}}/[4\pi\sqrt{\gamma^2(x-Vt)^2+y^2+z^2}\,].$$

d) The fields are found using $\boldsymbol{E}(\boldsymbol{r},t) = -\boldsymbol{\nabla}\psi(\boldsymbol{r},t) - \partial\boldsymbol{A}(\boldsymbol{r},t)/\partial t$ and $\boldsymbol{B}(\boldsymbol{r},t) = \boldsymbol{\nabla}\times\boldsymbol{A}(\boldsymbol{r},t)$, as follows:

$$\boldsymbol{E}(\boldsymbol{r},t) = (\gamma q/4\pi\varepsilon_o)[(x-Vt)\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}]/[\gamma^2(x-Vt)^2+y^2+z^2]^{3/2};$$
$$\boldsymbol{B}(\boldsymbol{r},t) = (\mu_o q\gamma V/4\pi)(-z\hat{\boldsymbol{y}} + y\hat{\boldsymbol{z}})/[\gamma^2(x-Vt)^2+y^2+z^2]^{3/2}.$$