Opti 501 **Solutions**

Problem 25) a) The standard Maxwell's equations are

$$\boldsymbol{\nabla}\cdot\boldsymbol{D}(\boldsymbol{r},t)=\rho_{\text{free}}(\boldsymbol{r},t), \tag{1a}$$

$$\boldsymbol{\nabla}\times\boldsymbol{H}(\boldsymbol{r},t)=\boldsymbol{J}_{\text{free}}(\boldsymbol{r},t)+\partial\boldsymbol{D}(\boldsymbol{r},t)/\partial t, \tag{1b}$$

$$\boldsymbol{\nabla}\times\boldsymbol{E}(\boldsymbol{r},t)=-\partial\boldsymbol{B}(\boldsymbol{r},t)/\partial t, \tag{1c}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{B}(\boldsymbol{r},t)=0. \tag{1d}$$

To eliminate $\boldsymbol{E}$ and $\boldsymbol{B}$, one need only modify the third and fourth equations, as follows:

Eq. (1c): $\varepsilon_o\boldsymbol{\nabla}\times\boldsymbol{E}(\boldsymbol{r},t)+\boldsymbol{\nabla}\times\boldsymbol{P}(\boldsymbol{r},t)=-\varepsilon_o\partial\boldsymbol{M}(\boldsymbol{r},t)/\partial t-\varepsilon_o\mu_o\partial\boldsymbol{H}(\boldsymbol{r},t)/\partial t+\boldsymbol{\nabla}\times\boldsymbol{P}(\boldsymbol{r},t)$

$$\Rightarrow\ \boldsymbol{\nabla}\times\boldsymbol{D}(\boldsymbol{r},t)=-\varepsilon_o[\partial\boldsymbol{M}(\boldsymbol{r},t)/\partial t-\varepsilon_o^{-1}\boldsymbol{\nabla}\times\boldsymbol{P}(\boldsymbol{r},t)]-\varepsilon_o\mu_o\partial\boldsymbol{H}(\boldsymbol{r},t)/\partial t. \tag{1c$'$}$$

Eq. (1d): $$\mu_o\boldsymbol{\nabla}\cdot\boldsymbol{H}(\boldsymbol{r},t)=-\boldsymbol{\nabla}\cdot\boldsymbol{M}(\boldsymbol{r},t). \tag{1d$'$}$$

b) Transforming the modified equations to the Fourier domain yields,

$$\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{D}(\boldsymbol{k},\omega)=\rho_{\text{free}}(\boldsymbol{k},\omega), \tag{2a}$$

$$\mathrm{i}\boldsymbol{k}\times\boldsymbol{H}(\boldsymbol{k},\omega)=\boldsymbol{J}_{\text{free}}(\boldsymbol{k},\omega)-\mathrm{i}\,\omega\boldsymbol{D}(\boldsymbol{k},\omega), \tag{2b}$$

$$\boldsymbol{k}\times\boldsymbol{D}(\boldsymbol{k},\omega)=\varepsilon_o\omega\boldsymbol{M}(\boldsymbol{k},\omega)+\boldsymbol{k}\times\boldsymbol{P}(\boldsymbol{k},\omega)+(\omega/c^2)\boldsymbol{H}(\boldsymbol{k},\omega), \tag{2c}$$

$$\mu_o\boldsymbol{k}\cdot\boldsymbol{H}(\boldsymbol{k},\omega)=-\boldsymbol{k}\cdot\boldsymbol{M}(\boldsymbol{k},\omega). \tag{2d}$$

c) Cross-multiplying "$-\mathrm{i}\boldsymbol{k}$" into Eq. (2b), one arrives at

$$\boldsymbol{k}\times[\boldsymbol{k}\times\boldsymbol{H}(\boldsymbol{k},\omega)]=-\mathrm{i}\boldsymbol{k}\times\boldsymbol{J}_{\text{free}}(\boldsymbol{k},\omega)-\omega\boldsymbol{k}\times\boldsymbol{D}(\boldsymbol{k},\omega). \tag{3}$$

Using the vector identity $\boldsymbol{k}\times(\boldsymbol{k}\times\boldsymbol{H})=(\boldsymbol{k}\cdot\boldsymbol{H})\boldsymbol{k}-k^2\boldsymbol{H}$ in the preceding equation, one obtains

$$[\boldsymbol{k}\cdot\boldsymbol{H}(\boldsymbol{k},\omega)]\boldsymbol{k}-k^2\boldsymbol{H}(\boldsymbol{k},\omega)=-\mathrm{i}\boldsymbol{k}\times\boldsymbol{J}_{\text{free}}(\boldsymbol{k},\omega)-\omega\boldsymbol{k}\times\boldsymbol{D}(\boldsymbol{k},\omega). \tag{4}$$

Substitution from Eqs. (2c) and (2d) into Eq. (4) then yields

$$-\mu_o^{-1}[\boldsymbol{k}\cdot\boldsymbol{M}(\boldsymbol{k},\omega)]\boldsymbol{k}-k^2\boldsymbol{H}(\boldsymbol{k},\omega)=-\mathrm{i}\boldsymbol{k}\times\boldsymbol{J}_{\text{free}}(\boldsymbol{k},\omega)-\varepsilon_o\omega^2\boldsymbol{M}(\boldsymbol{k},\omega)-\omega\boldsymbol{k}\times\boldsymbol{P}(\boldsymbol{k},\omega)-(\omega^2/c^2)\boldsymbol{H}(\boldsymbol{k},\omega)$$

$$\Rightarrow\ \boxed{\boldsymbol{H}(\boldsymbol{k},\omega)=\{\mathrm{i}\boldsymbol{k}\times\boldsymbol{J}_{\text{free}}(\boldsymbol{k},\omega)+\omega\boldsymbol{k}\times\boldsymbol{P}(\boldsymbol{k},\omega)+\varepsilon_o\omega^2\boldsymbol{M}(\boldsymbol{k},\omega)-\mu_o^{-1}[\boldsymbol{k}\cdot\boldsymbol{M}(\boldsymbol{k},\omega)]\boldsymbol{k}\}/(k^2-\omega^2/c^2).} \tag{5}$$

To determine $\boldsymbol{D}(\boldsymbol{k},\omega)$, proceed along similar lines, namely, cross-multiply $\boldsymbol{k}$ into Eq. (2c), use the vector identity $\boldsymbol{k}\times(\boldsymbol{k}\times\boldsymbol{D})=(\boldsymbol{k}\cdot\boldsymbol{D})\boldsymbol{k}-k^2\boldsymbol{D}$, then substitute from Eqs. (2a) and (2b) into the resulting equation to obtain

$$\boldsymbol{k}\times[\boldsymbol{k}\times\boldsymbol{D}(\boldsymbol{k},\omega)]=\varepsilon_o\omega\boldsymbol{k}\times\boldsymbol{M}(\boldsymbol{k},\omega)+\boldsymbol{k}\times[\boldsymbol{k}\times\boldsymbol{P}(\boldsymbol{k},\omega)]+(\omega/c^2)\boldsymbol{k}\times\boldsymbol{H}(\boldsymbol{k},\omega)$$

$$\Rightarrow\ [\boldsymbol{k}\cdot\boldsymbol{D}(\boldsymbol{k},\omega)]\boldsymbol{k}-k^2\boldsymbol{D}(\boldsymbol{k},\omega)=\varepsilon_o\omega\boldsymbol{k}\times\boldsymbol{M}(\boldsymbol{k},\omega)+[\boldsymbol{k}\cdot\boldsymbol{P}(\boldsymbol{k},\omega)]\boldsymbol{k}-k^2\boldsymbol{P}(\boldsymbol{k},\omega)$$
$$-\mathrm{i}(\omega/c^2)\boldsymbol{J}_{\text{free}}(\boldsymbol{k},\omega)-(\omega^2/c^2)\boldsymbol{D}(\boldsymbol{k},\omega)$$

$$\Rightarrow\ \boxed{\begin{aligned}\boldsymbol{D}(\boldsymbol{k},\omega)=\{&-\mathrm{i}\rho_{\text{free}}(\boldsymbol{k},\omega)\boldsymbol{k}+\mathrm{i}(\omega/c^2)\boldsymbol{J}_{\text{free}}(\boldsymbol{k},\omega)+k^2\boldsymbol{P}(\boldsymbol{k},\omega)-[\boldsymbol{k}\cdot\boldsymbol{P}(\boldsymbol{k},\omega)]\boldsymbol{k}\\&-\varepsilon_o\omega\boldsymbol{k}\times\boldsymbol{M}(\boldsymbol{k},\omega)\}/(k^2-\omega^2/c^2).\end{aligned}} \tag{6}$$