Problem 24

a) From problem 4 we have $\vec{E}(\vec{r}) = \frac{q_0}{4\pi\epsilon_0}\left(\frac{\hat{r}}{r^2}\right)$.

From problem 13 we have $\vec{H}(\vec{r}) = \frac{m_0}{4\pi\mu_0}\,\frac{3\cos\theta\,\hat{r}-\hat{z}}{r^3}$. Therefore,

$$\vec{S}(\vec{r}) = \vec{E}(\vec{r})\times\vec{H}(\vec{r}) = \frac{m_0 q_0}{16\pi^2\mu_0\epsilon_0}\,\frac{\hat{r}\times(3\cos\theta\,\hat{r}-\hat{z})}{r^5} = \frac{m_0 q_0 c^2}{16\pi^2 r^5}\sin\theta\,\hat{\phi}$$

b) Available energy density $= \frac{1}{2}\epsilon_0\vec{E}(\vec{r})\cdot\vec{E}(\vec{r}) + \frac{1}{2}\mu_0\vec{H}(\vec{r})\cdot\vec{H}(\vec{r}) \Rightarrow$

$$\mathcal{E}(\vec{r}) = \frac{1}{2}\epsilon_0\left(\frac{q_0}{4\pi\epsilon_0 r^2}\right)^2 + \frac{1}{2}\mu_0\left(\frac{m_0}{4\pi\mu_0 r^3}\right)^2(3\cos\theta\,\hat{r}-\hat{z})\cdot(3\cos\theta\,\hat{r}-\hat{z})$$

$$= \frac{q_0^2}{32\pi^2\epsilon_0 r^4} + \frac{m_0^2}{32\pi^2\mu_0 r^6}(9\cos^2\theta+1-6\cos^2\theta)$$

Available energy density within spherical shell of radius r, thickness $\Delta r =$

$$\int_{\theta=0}^{\pi}\mathcal{E}(\vec{r})\,2\pi r^2\sin\theta\,d\theta\,\Delta r = \frac{q_0^2\,\Delta r}{16\pi\epsilon_0 r^2}\int_0^{\pi}\sin\theta\,d\theta + \frac{m_0^2\,\Delta r}{16\pi\mu_0 r^4}\underbrace{\int_0^{\pi}(3\cos^2\theta+1)\sin\theta\,d\theta}_{=4}$$

$$= \frac{q_0^2\,\Delta r}{8\pi\epsilon_0 r^2} + \frac{m_0^2\,\Delta r}{4\pi\mu_0 r^4} = \frac{\Delta r}{8\pi}\left(\frac{q_0^2}{\epsilon_0 r^2} + \frac{2m_0^2}{\mu_0 r^4}\right)$$

$$= \frac{\Delta r}{8\pi}\left\{\underbrace{\left[\frac{q_0}{\sqrt{\epsilon_0}\,r} - \frac{\sqrt{2}\,m_0}{\sqrt{\mu_0}\,r^2}\right]^2}_{\text{always}\,\geq 0} + \frac{2\sqrt{2}\,q_0 m_0 c}{r^3}\right\}$$

Circulating energy around the point particle within a spherical shell of radius r and thickness $\Delta r =$

$$\int_{\theta=0}^{\pi}\frac{|\vec{S}(\vec{r})|}{c}\,2\pi r^2\sin\theta\,d\theta\,\Delta r = \frac{m_0 q_0 c\,\Delta r}{8\pi r^3}\underbrace{\int_0^{\pi}\sin^2\theta\,d\theta}_{=\pi/2} = \frac{m_0 q_0 c}{16 r^3}\Delta r$$

c) To prove that circulating energy (within the spherical shell) is less than the total available energy within the shell, we need to show that

$$\frac{m_0 q_0 c}{16 r^3}\Delta r \leq \frac{\sqrt{2}\,q_0 m_0 c}{4\pi r^3}\Delta r \Rightarrow \frac{1}{16} \leq \frac{\sqrt{2}}{4\pi} \Rightarrow \pi \leq 4\sqrt{2}$$ which is obviously valid. ✓