Problem 23) The scalar potential $\psi(\boldsymbol{r})$ of a uniformly-polarized sphere whose dipole-moment is given by $\boldsymbol{p}(t\!=\!0) = (4\pi R^3/3)\,P_o\,\hat{\boldsymbol{z}}$, was found in Problem 12 to be

$$\psi(r,\theta,\phi) = \begin{cases} \dfrac{P_o R^3 \cos\theta}{3\varepsilon_o r^2}; & r \ge R \\[2ex] \dfrac{P_o r\cos\theta}{3\varepsilon_o}; & r \le R \end{cases} \tag{1}$$

As the sphere rotates, its dipole moment $\boldsymbol{p}(t)$ deviates from the z-axis, assuming the direction of the arbitrary unit-vector $\hat{\boldsymbol{u}} = (\sin\eta)\hat{\boldsymbol{y}} + (\cos\eta)\hat{\boldsymbol{z}}$ in the yz-plane. Denoting the observation point by $\boldsymbol{r} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}$, the angle θ between $\boldsymbol{p}(t)$ and the observation point will be given by

$$\cos\theta = \hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{u}} = \frac{y\sin\eta + z\cos\eta}{\sqrt{x^2+y^2+z^2}}. \tag{2}$$

The scalar potential of the polarized sphere may thus be written as follows:

$$\psi(x,y,z) = \begin{cases} \dfrac{P_o R^3 (y\sin\eta + z\cos\eta)}{3\varepsilon_o (x^2+y^2+z^2)^{3/2}}; & r \ge R \\[2ex] \dfrac{P_o (y\sin\eta + z\cos\eta)}{3\varepsilon_o}; & r \le R \end{cases} \tag{3}$$

As we are interested only in the z-component of the E-field of the polarized sphere, we write

$$E_z^{\text{dipole}}(x,y,z) = -\frac{\partial\psi}{\partial z} = \begin{cases} -\dfrac{P_o R^3}{\varepsilon_o}\left(\dfrac{\cos\eta}{3(x^2+y^2+z^2)^{3/2}} - \dfrac{yz\sin\eta + z^2\cos\eta}{(x^2+y^2+z^2)^{5/2}}\right); & r \ge R \\[2ex] -\dfrac{P_o\cos\eta}{3\varepsilon_o}; & r \le R \end{cases} \tag{4}$$

The time-dependent contribution to the total energy of the system is the integral of $\varepsilon_o E_o E_z^{\text{dipole}}$ over the volume between the parallel-plates. Now, the term containing "$yz\sin\eta$" in Eq.(4) does not contribute to the total energy, because for every $\boldsymbol{r} = (x,y,z)$ there will be an $\boldsymbol{r}' = (x,-y,z)$, which cancels its contribution. As for the remaining terms, fixing z at z_o, where $|z_o| > R$, and integrating over the xy- plane, we find

$$\iint_{-\infty}^{\infty}\left(\frac{\cos\eta}{3(x^2+y^2+z_o^2)^{3/2}} - \frac{yz_o\sin\eta + z_o^2\cos\eta}{(x^2+y^2+z_o^2)^{5/2}}\right)\mathrm{d}x\mathrm{d}y$$

$$= \cos\eta \iint_{-\infty}^{\infty}\left(\frac{1}{3(x^2+y^2+z_o^2)^{3/2}} - \frac{z_o^2}{(x^2+y^2+z_o^2)^{5/2}}\right)\mathrm{d}x\mathrm{d}y$$

$$= \cos\eta \int_0^\infty \left(\frac{1}{3(\rho^2+z_o^2)^{3/2}} - \frac{z_o^2}{(\rho^2+z_o^2)^{5/2}} \right) 2\pi\rho\,\mathrm{d}\rho$$

$$= 2\pi\cos\eta \left(-\tfrac{1}{3}(\rho^2+z_o^2)^{-1/2}\Big|_0^\infty + \tfrac{1}{3} z_o^2 (\rho^2+z_o^2)^{-3/2}\Big|_0^\infty \right)$$

$$= 2\pi\cos\eta \left(\frac{1}{3z_o} - \frac{z_o^2}{3z_o^3} \right) = 0. \qquad (5)$$

The only contribution to the time-dependent E-field energy of the system thus comes from the integral of $\varepsilon_o E_o E_z^{\text{dipole}}$ over the region $-R < z < R$. Inside the sphere, the volume integration of E_z^{dipole} yields $-4\pi R^3 P_o \cos\eta/(9\varepsilon_o)$; see Eq.(4). The integral outside the sphere is given by

$$\int_{z=-R}^{R} \int_{\rho=\sqrt{R^2-z^2}}^{\infty} 2\pi\rho E_z^{\text{dipole}}(\rho, z)\,\mathrm{d}\rho$$

$$= \frac{2\pi P_o R^3 \cos\eta}{\varepsilon_o} \int_{z=-R}^{R} \int_{\rho=\sqrt{R^2-z^2}}^{\infty} \left(\frac{z^2\rho}{(\rho^2+z^2)^{5/2}} - \frac{\rho}{3(\rho^2+z^2)^{3/2}} \right) \mathrm{d}\rho\,\mathrm{d}z$$

$$= \frac{2\pi P_o R^3 \cos\eta}{\varepsilon_o} \int_{z=-R}^{R} \left[\tfrac{1}{3} z^2 (\rho^2+z^2)^{-3/2} - \tfrac{1}{3}(\rho^2+z^2)^{-1/2} \right]_{\rho=\sqrt{R^2-z^2}}^{\infty} \mathrm{d}z$$

$$= \frac{2\pi P_o R^3 \cos\eta}{3\varepsilon_o} \int_{z=-R}^{R} [(z^2/R^3) - (1/R)]\,\mathrm{d}z$$

$$= -\frac{8\pi P_o R^3 \cos\eta}{9\varepsilon_o}. \qquad (6)$$

Adding the contributions to the total E-field energy of the regions inside and outside the sphere, we find

$$\int_{x=-\infty}^{\infty} \int_{y=-\infty}^{\infty} \int_{z=-d/2}^{d/2} \varepsilon_o E_o E_z^{\text{dipole}} \mathrm{d}x\mathrm{d}y\mathrm{d}z = -(4\pi R^3 P_o/3)\, E_o \cos\eta. \qquad (7)$$

The time-rate-of-change of the total E-field energy of the system is, therefore, given by

$$\frac{\partial}{\partial t} \int_{-\infty}^{\infty} \mathcal{E}(\boldsymbol{r}, t)\,\mathrm{d}\boldsymbol{r} = (4\pi R^3 P_o/3)\, E_o \sin[\eta(t)] \frac{\partial \eta(t)}{\partial t}. \qquad (8)$$

Thus, as η increases from 0 to π, the energy stored in the E-field rises continually. At the same time, since $\boldsymbol{p}(t) = (4\pi R^3 P_o/3)\{\sin[\eta(t)]\,\hat{\boldsymbol{y}} + \cos[\eta(t)]\,\hat{\boldsymbol{z}}\}$, we will have

$$\boldsymbol{E}_o \cdot \mathrm{d}\boldsymbol{p}(t)/\mathrm{d}t = -(4\pi R^3 P_o/3)\, E_o \sin[\eta(t)] \frac{\partial \eta(t)}{\partial t}. \qquad (9)$$

The negative sign of $\boldsymbol{E}_o \cdot \mathrm{d}\boldsymbol{p}(t)/\mathrm{d}t$ in the above equation indicates that, as η increases from 0 to π, the dipole gives energy to the system. This, of course, is the same energy that is being stored in the E-field throughout the surrounding space. Needless to say, if the dipole reverses course and η begins to decrease, the stored energy will go down, as the dipole begins to take up energy from the surrounding E-field.
