Problem 22)

Inside the sphere — Outside the sphere

a) $$\int_{-\infty}^{\infty}\mathcal{E}(\boldsymbol{r})\mathrm{d}\boldsymbol{r}=\tfrac{1}{2}\varepsilon_o\int_{-\infty}^{\infty}|\boldsymbol{E}|^2\mathrm{d}\boldsymbol{r}=\underbrace{\frac{P_o^2(4\pi R^3/3)}{18\varepsilon_o}}_{\text{Inside the sphere}}+\underbrace{\frac{P_o^2R^6}{18\varepsilon_o}\int_{r=R}^{\infty}\int_{\theta=0}^{\pi}\frac{4\cos^2\theta+\sin^2\theta}{r^6}2\pi r^2\sin\theta\,\mathrm{d}\theta\mathrm{d}\boldsymbol{r}}_{\text{Outside the sphere}}$$

$$=\frac{P_o^2(4\pi R^3/3)}{18\varepsilon_o}+\frac{2\pi P_o^2R^6}{18\varepsilon_o}\int_{r=R}^{\infty}\int_{\theta=0}^{\pi}\frac{(3\cos^2\theta+1)\sin\theta}{r^4}\,\mathrm{d}\theta\mathrm{d}\boldsymbol{r}$$

$$=\frac{P_o^2(4\pi R^3/3)}{18\varepsilon_o}+\frac{2\pi P_o^2R^6}{18\varepsilon_o}(-\frac{1}{3r^3})\Big|_{r=R}^{\infty}(-\cos^3\theta-\cos\theta)\Big|_0^{\pi}$$

$$=\frac{P_o^2(4\pi R^3/3)}{18\varepsilon_o}+\frac{2P_o^2(4\pi R^3/3)}{18\varepsilon_o}$$

$$=\frac{P_o^2(4\pi R^3/3)}{6\varepsilon_o}$$

The E-field energy is thus seen to be divided between the inside and outside regions of the sphere, with the inside region containing one-third of the total energy. Denoting the dipole moment by $\boldsymbol{p}=(4\pi R^3/3)P_o\hat{\boldsymbol{z}}$, the total energy may also be expressed as $p^2/(8\pi\varepsilon_o R^3)$. This so-called self-energy of the dipole increases without bound as the sphere radius shrinks while $\boldsymbol{p}$ is kept constant.

b) Let the separation between the centers of the two spheres be z, where $0\le z\le d$. The polarization will then be $\boldsymbol{P}_o=\rho_o z\hat{\boldsymbol{z}}$, and the (uniform) E-field acting on the charges will be $\boldsymbol{E}=-(\boldsymbol{P}_o/3\varepsilon_o)=-(\rho_o z/3\varepsilon_o)\hat{\boldsymbol{z}}$. Let the sphere of positive charge be the one that is being pulled away, while the other sphere is kept in place. The total charge of the sphere, $Q=(4\pi R^3/3)\rho_o$, multiplied by the E-field, then gives the required force as $\boldsymbol{F}=-Q\boldsymbol{E}=(4\pi\rho_o^2R^3/9\varepsilon_o)z\hat{\boldsymbol{z}}$. The integral of this force from $z=0$ to d then yields the total energy needed to pull the charges apart, as follows:

$$\int_{z=0}^{d}F(z)\mathrm{d}z=\frac{4\pi\rho_o^2R^3}{9\varepsilon_o}\int_{z=0}^{d}z\,\mathrm{d}z=\frac{4\pi\rho_o^2d^2R^3}{18\varepsilon_o}=\frac{P_o^2(4\pi R^3/3)}{6\varepsilon_o}.$$

The final expression is the same as that obtained in part (a) by integrating the E-field energy density of the dipole over the entire space.