Problem 21)

a) The superposition principle holds here, so we can add the potentials produced by the two plates. The scalar potential is zero for both plates, therefore, $\psi(\vec{r},t) = \psi_1(\vec{r},t) + \psi_2(\vec{r},t) = 0$. As for the vector potential, we have (see problem 21):

$$\vec{A}(\vec{r},t) = \vec{A}_1(\vec{r},t) + \vec{A}_2(\vec{r},t) = \frac{1}{2}\frac{Z_0 J_{s0}\hat{z}}{\omega_0}\left\{\sin\left[\omega_0\left(t - \frac{|y|}{c}\right)\right] + \sin\left[\omega_0\left(t - \frac{|y-d|}{c}\right)\right]\right\}$$

We would like to set $\vec{A}(\vec{r},t) = 0$ outside the cavity, where $y < 0$ or $y > d$. Note that, in the region $y < 0$ we can write $|y-d|$ as $|y| + d$, whereas for $y > d$ we have $|y-d| = |y| - d$. If we set $\omega_0 d/c = (2m+1)\pi$, where m is any integer, we will have:

$$\sin\left[\omega_0\left(t - \frac{|y-d|}{c}\right)\right] = \sin\left[\omega_0\left(t - \frac{|y|}{c}\right) \pm (2m+1)\pi\right] = -\sin\left[\omega_0\left(t - \frac{|y|}{c}\right)\right] \leftarrow \text{outside cavity}$$

This will ensure that $\vec{A}(\vec{r},t) = 0$ outside the cavity. For resonance condition, therefore, the required distance d between the plates is

$$d = (2m+1)\pi c/\omega_0 \quad \leftarrow \text{any integer } m = 0, 1, 2, \ldots \text{ will do.}$$

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b) Between the plates $|y| = y$, whereas $|y-d| = d - y$. We have:

$$\vec{A}(\vec{r},t) = \frac{Z_0 J_{s0}\hat{z}}{2\omega_0}\left\{\sin[\omega_0(t - y/c)] + \sin[\omega_0(t + y/c) - (2m+1)\pi]\right\}$$

$$= \frac{Z_0 J_{s0}\hat{z}}{2\omega_0}\left\{\sin[\omega_0(t - y/c)] - \sin[\omega_0(t + y/c)]\right\} \Rightarrow$$

$$\vec{A}(\vec{r},t) = -\frac{Z_0 J_{s0}\hat{z}}{\omega_0}\sin(\omega_0 y/c)\cos(\omega_0 t) \quad \leftarrow \text{within the cavity}$$

$$\vec{E}(\vec{r},t) = -\vec{\nabla}\psi - \frac{\partial\vec{A}}{\partial t} = -\frac{\partial\vec{A}}{\partial t} = -Z_0 J_{s0}\hat{z}\sin(\omega_0 y/c)\sin(\omega_0 t)$$

Note that at $y = 0$ and $y = d$, we have $\sin(\omega_0 y/c) = 0$, which means that the net E-field acting on the plates is zero. The oscillations, therefore, sustain themselves. In other words, the radiation from one plate provides the necessary driving E-field for the other plate, and vice-versa.

$$\vec{H}(\vec{r},t) = \frac{1}{\mu_0}\vec{\nabla}\times\vec{A}(\vec{r},t) = \frac{1}{\mu_0}\frac{\partial A_z}{\partial y}\hat{x} = -\frac{1}{\mu_0}\frac{Z_0 J_{s0}}{\cancel{\omega_0}}\left(\frac{\cancel{\omega_0}}{c}\right)\cos(\omega_0 y/c)\cos(\omega_0 t)\hat{x} \Rightarrow$$

$$\vec{H}(\vec{r},t) = -J_{s0}\cos(\omega_0 y/c)\cos(\omega_0 t)\hat{x}$$

Note that at the surface of the mirrors, where $y=0$ and $y=d$, the magnetic field $\vec{H}$ is equal in magnitude to the surface current density $\vec{J} = J_{s0}\cos(\omega_0 t)$, while its direction is such that the requisite boundary condition is satisfied.

c) $\vec{S}(\vec{r},t) = \vec{E}(\vec{r},t)\times\vec{H}(\vec{r},t) = Z_0 J_{s0}^2 \sin(\omega_0 y/c)\cos(\omega_0 y/c)\sin(\omega_0 t)\cos(\omega_0 t)(\hat{z}\times\hat{x})$

$$\Rightarrow \vec{S}(\vec{r},t) = \frac{1}{4} Z_0 J_{s0}^2 \sin(2\omega_0 y/c)\sin(2\omega_0 t)\hat{y}$$ ← inside the cavity

The energy flux is directed along the y-axis. Its time-dependence is given by $\sin(2\omega_0 t)$, which means that the time-averaged rate of flow of energy is zero. The local energy shifts to the right and then to the left periodically, but it does not go anywhere on average. The dependence on y is given by $\sin(2\omega_0 y/c)$; when $\sin(2\omega_0 y/c)=0$, e.g., on the mirror surfaces, or at the nodes of the standing wave within the cavity, there is no flow of energy at all; the energy only shifts locally within the space bounded by adjacent nodes of the standing wave. Note that, when $\cos(\omega_0 t)=0$, all the energy is in the $\vec{E}$-field. A quarter of a period later, when $\sin(\omega_0 t)$ becomes zero, all the energy is transferred to the H-field. Conservation of energy then demands that the $\vec{E}$-field and $\vec{H}$-field energies be equal, which is something that is readily verified.