Problem 20)

$$\vec{\nabla}\cdot\vec{J}+\frac{\partial\rho}{\partial t}=0 \Rightarrow \frac{\partial\rho}{\partial t}=-\frac{\partial J_z}{\partial z}=-J_{s_0}\delta(y)e^{\omega'' t}K\sin(\omega' t-Kz) \Rightarrow$$

$$\frac{\partial\rho}{\partial t}=-KJ_{s_0}\delta(y)\frac{e^{-iKz}e^{+i(\omega'-i\omega'')t}-e^{+iKz}e^{-i(\omega'+i\omega'')t}}{2i} \Rightarrow$$

$$\rho(\vec{r},t)=\tfrac{1}{2}KJ_{s_0}\delta(y)\left[\frac{e^{iKz}e^{-i\omega_0 t}}{\omega_0}+\frac{e^{-iKz}e^{i\omega_0^* t}}{\omega_0^*}\right]$$

$$\rho(\vec{k},t)=\int_{-\infty}^{\infty}\rho(\vec{r},t)e^{-i\vec{k}\cdot\vec{r}}d\vec{r}=2\pi^2KJ_{s_0}\delta(k_x)\left[\frac{\delta(k_z-K)e^{-i\omega_0 t}}{\omega_0}+\frac{\delta(k_z+K)e^{+i\omega_0^* t}}{\omega_0^*}\right]$$

a) We find the scalar potential $\psi(\vec{k},t)$ and its inverse Fourier transform $\psi(\vec{r},t)$ first for the term containing ω_0, then for the term containing ω_0^*, and finally add them up after allowing $\omega''\to 0$. (See Problem 40 for a discussion of the choice of sign.)

$$\psi_1(\vec{r},t)=\frac{1}{(2\pi)^3\epsilon_0}\int_{-\infty}^{\infty}\frac{\rho(\vec{k},t)}{k^2-(\omega_0/c)^2}e^{+i\vec{k}\cdot\vec{r}}d\vec{k}=\frac{KJ_{s_0}}{4\pi\epsilon_0\omega_0}e^{-i\omega_0 t}e^{iKz}\int_{-\infty}^{\infty}\frac{e^{ik_y y}}{k_y^2+K^2-(\omega_0/c)^2}dk_y$$

$$=\begin{cases}\dfrac{KJ_{s_0}}{4\epsilon_0\omega_0}\dfrac{\exp\left(-i\omega_0 t+iKz+i\sqrt{(\omega_0/c)^2-K^2}\,|y|\right)}{-i\sqrt{(\omega_0/c)^2-K^2}}; & K<\omega_0/c\\[2ex] \dfrac{KJ_{s_0}}{4\epsilon_0\omega_0}\dfrac{\exp(-i\omega_0 t+iKz)\exp\left(-\sqrt{K^2-(\omega_0/c)^2}\,|y|\right)}{\sqrt{K^2-(\omega_0/c)^2}}; & K>\omega_0/c\end{cases}$$

b) Similarly, we find $\psi(\vec{r},t)$ for the term containing ω_0^* and allow $\omega''\to 0$.

$$\psi_2(\vec{r},t)=\begin{cases}\dfrac{KJ_{s_0}}{4\epsilon_0\omega_0}\dfrac{\exp\left(i\omega_0 t-iKz-i\sqrt{(\omega_0/c)^2-K^2}\,|y|\right)}{i\sqrt{(\omega_0/c)^2-K^2}}; & K<\omega_0/c\\[2ex] \dfrac{KJ_{s_0}}{4\epsilon_0\omega_0}\dfrac{\exp(i\omega_0 t-iKz)\exp\left(-\sqrt{K^2-(\omega_0/c)^2}\,|y|\right)}{\sqrt{K^2-(\omega_0/c)^2}}; & K>\omega_0/c\end{cases}$$

Adding the above expressions yields the final form of the scalar potential $\psi(\vec{r},t)$.

$$\psi(\vec{r},t)=\psi_1+\psi_2=\begin{cases}\dfrac{KJ_{s_0}}{2\epsilon_0\omega_0\sqrt{(\omega_0/c)^2-K^2}}\sin\left[\omega_0 t-Kz-\sqrt{(\omega_0/c)^2-K^2}\,|y|\right]; & K<\omega_0/c\\[2ex] \dfrac{KJ_{s_0}}{2\epsilon_0\omega_0\sqrt{K^2-(\omega_0/c)^2}}\exp\left(-\sqrt{K^2-(\omega_0/c)^2}\,|y|\right)\cos(\omega_0 t-Kz); & K>\omega_0/c\end{cases}$$

A similar procedure is followed for determining the vector potential $\vec{A}(\vec{r},t)$.

$$\vec{J}(\vec{r},t)=\tfrac{1}{2}J_{s_0}\delta(y)\hat{z}\left[\exp(-i\omega_0 t+Kz)+\exp(i\omega_0^* t-Kz)\right]$$

$$\vec{J}(\vec{k},t) = \int_{-\infty}^{\infty} \vec{J}(\vec{r},t)\, e^{-i\vec{k}\cdot\vec{r}}\, d\vec{r} = 2\pi^2 J_{s_0}\, \delta(k_x)\, \hat{z}\left[\delta(k_z-\kappa)\, e^{-i\omega_0 t} + \delta(k_z+\kappa)\, e^{+i\omega_0^* t}\right]$$

$$\vec{A}_1(\vec{r},t) = \frac{\mu_0}{(2\pi)^3}\int_{-\infty}^{\infty}\frac{\vec{J}_1(\vec{k},t)}{k^2-(\omega_0/c)^2}\, e^{+i\vec{k}\cdot\vec{r}}\, d\vec{k} = \frac{\mu_0 J_{s_0}\hat{z}}{4\pi}\, e^{-i\omega_0 t}\, e^{i\kappa z}\int_{-\infty}^{\infty}\frac{e^{ik_y y}}{k_y^2+\kappa^2-(\omega_0/c)^2}\, dk_y$$

$$= \begin{cases} \frac{1}{4}\mu_0 J_{s_0}\hat{z}\; \dfrac{\exp\left(-i\omega_0 t + i\kappa z + i\sqrt{(\omega_0/c)^2-\kappa^2}\,|y|\right)}{-i\sqrt{(\omega_0/c)^2-\kappa^2}}\,; & \kappa < \omega_0/c \\[2ex] \frac{1}{4}\mu_0 J_{s_0}\hat{z}\; \dfrac{\exp(-i\omega_0 t + i\kappa z)\exp\left(-\sqrt{\kappa^2-(\omega_0/c)^2}\,|y|\right)}{\sqrt{\kappa^2-(\omega_0/c)^2}}\,; & \kappa > \dfrac{\omega_0}{c} \end{cases}$$

A similar expression is found for the term containing ω_0^*. When the two expressions are added together, we find:

$$\vec{A}(\vec{r},t) = \vec{A}_1(\vec{r},t) + \vec{A}_2(\vec{r},t) = \begin{cases} \dfrac{\mu_0 J_{s_0}\hat{z}}{2\sqrt{(\omega_0/c)^2-\kappa^2}}\, \sin\left[\omega_0 t - \kappa z - \sqrt{(\omega_0/c)^2-\kappa^2}\,|y|\right]; & \kappa < \dfrac{\omega_0}{c} \\[2ex] \dfrac{\mu_0 J_{s_0}\hat{z}}{2\sqrt{\kappa^2-(\omega_0/c)^2}}\, \exp\left(-\sqrt{\kappa^2-(\omega_0/c)^2}\,|y|\right)\cos(\omega_0 t - \kappa z); & \kappa > \dfrac{\omega_0}{c} \end{cases}$$

$$\vec{E}(\vec{r},t) = -\vec{\nabla}\psi - \frac{\partial \vec{A}}{\partial t} = -\frac{\partial\psi}{\partial y}\hat{y} - \left(\frac{\partial\psi}{\partial z} + \frac{\partial A_z}{\partial t}\right)\hat{z} \;\Rightarrow$$

$$E_y(\vec{r},t) = -\frac{\partial\psi}{\partial y} = \begin{cases} \dfrac{\kappa J_{s_0}\,\mathrm{Sign}(y)}{2\epsilon_0\omega_0}\cos\left[\omega_0 t - \kappa z - \sqrt{(\omega_0/c)^2-\kappa^2}\,|y|\right]; & \kappa < \omega_0/c \\[2ex] \dfrac{\kappa J_{s_0}\,\mathrm{Sign}(y)}{2\epsilon_0\omega_0}\exp\left(-\sqrt{\kappa^2-(\omega_0/c)^2}\,|y|\right)\cos(\omega_0 t - \kappa z); & \kappa > \omega_0/c. \end{cases}$$

$$E_z(\vec{r},t) = -\left(\frac{\partial\psi}{\partial z} + \frac{\partial A_z}{\partial t}\right) = \begin{cases} -\dfrac{J_{s_0}\sqrt{(\omega_0/c)^2-\kappa^2}}{2\epsilon_0\omega_0}\cos\left[\omega_0 t - \kappa z - \sqrt{(\omega_0/c)^2-\kappa^2}\,|y|\right]; & \kappa < \omega_0/c \\[2ex] -\dfrac{J_{s_0}\sqrt{\kappa^2-(\omega_0/c)^2}}{2\epsilon_0\omega_0}\exp\left(-\sqrt{\kappa^2-(\omega_0/c)^2}\,|y|\right)\sin(\omega_0 t - \kappa z); & \kappa > \omega_0/c \end{cases}$$

$$\vec{H}(\vec{r},t) = \frac{1}{\mu_0}\vec{\nabla}\times\vec{A}(\vec{r},t) = \frac{1}{\mu_0}\frac{\partial A_z}{\partial y}\hat{x} \;\Rightarrow$$

$$H_x(\vec{r},t) = \begin{cases} -\frac{1}{2}J_{s_0}\,\mathrm{Sign}(y)\cos\left[\omega_0 t - \kappa z - \sqrt{(\omega_0/c)^2-\kappa^2}\,|y|\right]; & \kappa < \omega_0/c \\[1ex] -\frac{1}{2}J_{s_0}\,\mathrm{Sign}(y)\exp\left(-\sqrt{\kappa^2-(\omega_0/c)^2}\,|y|\right)\cos(\omega_0 t - \kappa z); & \kappa > \omega_0/c \end{cases}$$

Note that E_y and H_x satisfy the boundary conditions at $y=0$.

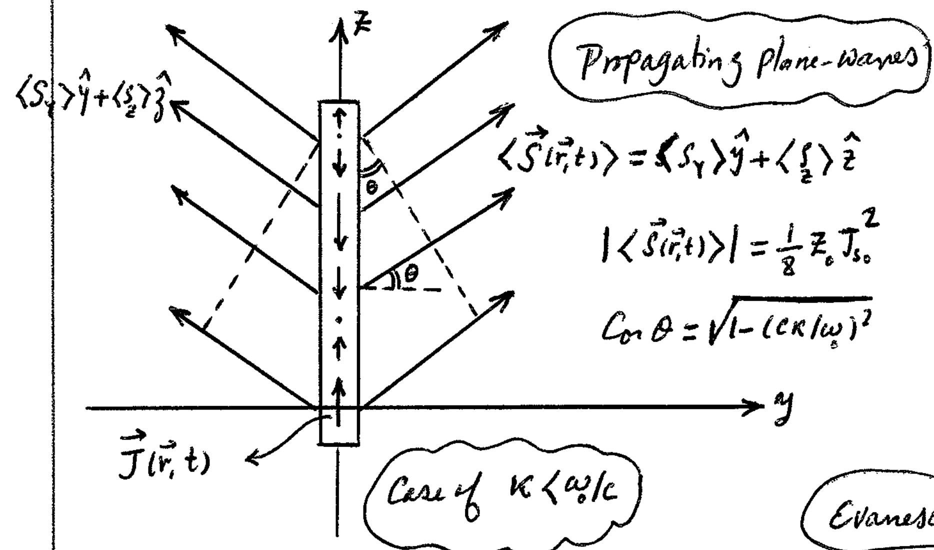
$$\vec{S}(\vec{r},t) = \vec{E}(\vec{r},t)\times\vec{H}(\vec{r},t) = (E_z H_x)\hat{y} - (E_y H_x)\hat{z} \quad \Rightarrow$$

$$S_y(\vec{r},t) = \begin{cases} \frac{1}{4} Z_0 J_{s0}^2 \,\mathrm{sign}(y)\sqrt{1-(c\kappa/\omega_0)^2}\, \cos^2\left[\omega_0 t - \kappa z - \sqrt{(\omega_0/c)^2-\kappa^2}\,|y|\right]; & \kappa < \omega_0/c \\ \frac{1}{4} Z_0 J_{s0}^2 \,\mathrm{sign}(y)\sqrt{(c\kappa/\omega_0)^2-1}\, \exp\left(-2\sqrt{\kappa^2-(\omega_0/c)^2}\,|y|\right) \sin(\omega_0 t-\kappa z)\cos(\omega_0 t - \kappa z); & \kappa > \omega_0/c \end{cases}$$

$$\Rightarrow \langle S_y(\vec{r},t)\rangle = \begin{cases} \frac{1}{8} Z_0 J_{s0}^2 \,\mathrm{sign}(y)\sqrt{1-(c\kappa/\omega_0)^2}; & \kappa < \omega_0/c \\ 0; & \kappa > \omega_0/c \end{cases}$$

$$S_z(\vec{r},t) = -E_y H_x = \begin{cases} \frac{1}{4} Z_0 J_{s0}^2 (c\kappa/\omega_0)\cos^2\left[\omega_0 t - \kappa z - \sqrt{(\omega_0/c)^2-\kappa^2}\,|y|\right]; & \kappa < \omega_0/c \\ \frac{1}{4} Z_0 J_{s0}^2 (c\kappa/\omega_0)\exp\left(-2\sqrt{\kappa^2-(\omega_0/c)^2}\,|y|\right)\cos^2(\omega_0 t - \kappa z); & \kappa > \omega_0/c \end{cases}$$

$$\Rightarrow \langle S_z(\vec{r},t)\rangle = \begin{cases} \frac{1}{8} Z_0 J_{s0}^2 (c\kappa/\omega_0); & \kappa < \omega_0/c \\ \frac{1}{8} Z_0 J_{s0}^2 (c\kappa/\omega_0)\exp\left(-2\sqrt{\kappa^2-(\omega_0/c)^2}\,|y|\right); & \kappa > \omega_0/c. \end{cases}$$



$$\langle \vec{S}(\vec{r},t)\rangle = \langle S_y\rangle\hat{y} + \langle S_z\rangle\hat{z}$$

$$|\langle \vec{S}(\vec{r},t)\rangle| = \tfrac{1}{8} Z_0 J_{s0}^2$$

$$\cos\theta = \sqrt{1-(c\kappa/\omega_0)^2}$$