Problem 19)

$$\vec J(\vec r,t)=J_{s_0}\,\delta(y)\,e^{\omega'' t}\,\frac{e^{i(\omega' t-\kappa x)}+e^{-i(\omega' t-\kappa x)}}{2}\,\hat z$$

$$=\tfrac12 J_{s_0}\delta(y)\hat z\left\{e^{-i\kappa x}e^{+i(\omega'-i\omega'')t}+e^{+i\kappa x}e^{-i(\omega'+i\omega'')t}\right\}$$

$$=\tfrac12 J_{s_0}\delta(y)\hat z\left\{e^{i\kappa x}e^{-i\omega_0 t}+e^{-i\kappa x}e^{+i\omega_0^* t}\right\}\quad\leftarrow \omega_0=\omega'+i\omega''$$

$$\Rightarrow \vec J(\vec k,t)=\int_{-\infty}^{\infty}\vec J(\vec r,t)\,e^{-i\vec k\cdot\vec r}\,d\vec r=\tfrac12 J_{s_0}\hat z\left\{e^{-i\omega_0 t}\int_{-\infty}^{\infty}\delta(y)\,e^{i\kappa x}\,e^{-i(k_x x+k_y y+k_z z)}\,dx\,dy\,dz\right.$$
$$\left.+\,e^{i\omega_0^* t}\int_{-\infty}^{\infty}\delta(y)\,e^{-i\kappa x}\,e^{-i(k_x x+k_y y+k_z z)}\,dx\,dy\,dz\right\}$$

$$\Rightarrow \vec J(\vec k,t)=2\pi^2 J_{s_0}\,\delta(k_z)\,\hat z\left\{\delta(k_x-\kappa)\,e^{-i\omega_0 t}+\delta(k_x+\kappa)\,e^{+i\omega_0^* t}\right\}$$

We find the vector potential $\vec A(\vec r,t)$ first for the term containing $e^{-i\omega_0 t}$, then for the term containing $e^{+i\omega_0^* t}$, then add the results.

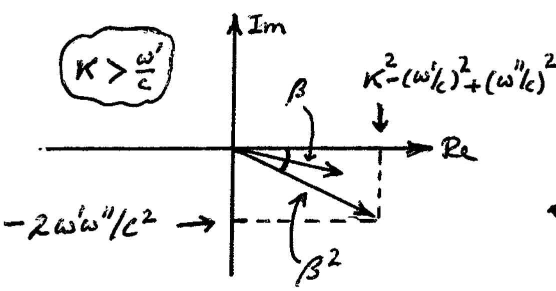
a) For the term containing $e^{-i\omega_0 t}$: $\vec A(\vec k,t)=\dfrac{2\pi^2\mu_0 J_{s_0}\,\delta(k_z)\,\delta(k_x-\kappa)\,\hat z}{k^2-(\omega_0/c)^2}\,e^{-i\omega_0 t}\ \Rightarrow$

$$\vec A(\vec r,t)=\frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}\vec A(\vec k,t)\,e^{+i\vec k\cdot\vec r}\,d\vec k=\frac{\mu_0}{4\pi}J_{s_0}\hat z\,e^{-i\omega_0 t}\int_{-\infty}^{\infty}\frac{\delta(k_z)\,\delta(k_x-\kappa)\,e^{i\vec k\cdot\vec r}}{(k_x^2+k_y^2+k_z^2)-(\omega_0/c)^2}\,d\vec k$$

$$=\frac{\mu_0 J_{s_0}\hat z}{4\pi}\,e^{i(\kappa x-\omega_0 t)}\int_{-\infty}^{\infty}\frac{e^{i k_y y}}{k_y^2+\kappa^2-(\omega_0/c)^2}\,dk_y$$

Now, $\beta^2=\kappa^2-(\omega_0/c)^2=\kappa^2-\left(\frac{\omega'+i\omega''}{c}\right)^2=\left[\kappa^2-(\omega'/c)^2+(\omega''/c)^2\right]-2i\left(\frac{\omega'\omega''}{c^2}\right);\quad \omega'>0,\ 0<\omega''\ll 1.$

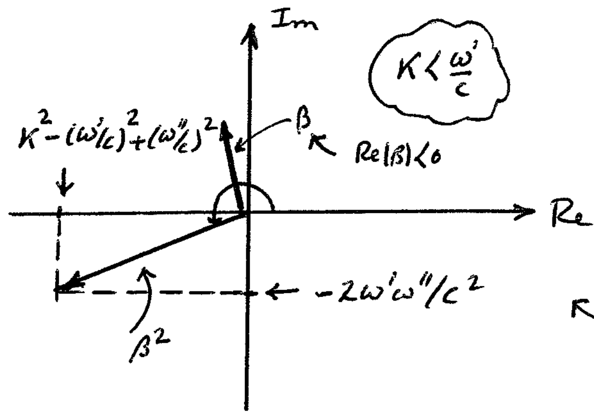
$\kappa>\frac{\omega'}{c}$



Note that if $\beta^2=|\beta|^2e^{i\phi}\Rightarrow\beta=|\beta|e^{i\phi/2}$ and $\beta=|\beta|e^{i(\phi+2\pi)/2}=|\beta|e^{i\pi}e^{i\phi/2}=-|\beta|e^{i\phi/2}$. Of the two possible answers for β we must choose the one which has a positive real value, as shown in the figure.

Therefore, in the case of $\kappa>\omega'/c$, in the limit when $\omega''\to0$, we'll have:

$\beta=\sqrt{\kappa^2-(\omega'/c)^2}=\sqrt{\kappa^2-(\omega_0/c)^2}.$



In the case of $K < \omega'/c$, we have $\beta^2 = |\beta|^2 e^{i\phi}$ with $\phi > 180°$. The β shown in the figure has a negative real part, thus the correct choice for β will be $-i\sqrt{(\omega_0/c)^2 - K^2}$.

The integral is thus evaluated as follows (with $\omega'' \to 0$):

$$\vec{A}(\vec{r},t) = \frac{\mu_0 J_{s0}\hat{z}}{4\pi} e^{i(Kx-\omega_0 t)} \begin{cases} \dfrac{\pi}{-i\sqrt{(\omega_0/c)^2-K^2}} e^{+i\sqrt{(\omega_0/c)^2-K^2}\,|y|} ; & K < \omega_0/c \\ \dfrac{\pi}{\sqrt{K^2-(\omega_0/c)^2}} e^{-\sqrt{K^2-(\omega_0/c)^2}\,|y|} ; & K > \dfrac{\omega_0}{c} \end{cases}$$

b) for the term containing $e^{+i\omega_0^* t}$ we follow a similar procedure. We'll have:

$$\vec{A}(\vec{r},t) = \frac{\mu_0 J_{s0}\hat{z}}{4\pi} e^{+i(-Kx+\omega_0^* t)} \int_{-\infty}^{\infty} \frac{e^{ik_y y}}{k_y^2 + K^2 - (\omega_0^*/c)^2} dk_y$$

$\omega'' \to 0$

$$= \frac{\mu_0 J_{s0}\hat{z}}{4\pi} e^{i(-Kx+\omega_0 t)} \begin{cases} \dfrac{\pi}{+i\sqrt{(\omega_0/c)^2-K^2}} e^{-i\sqrt{(\omega_0/c)^2-K^2}\,|y|} ; & K < \omega_0/c \\ \dfrac{\pi}{\sqrt{K^2-(\omega_0/c)^2}} e^{-\sqrt{K^2-(\omega_0/c)^2}\,|y|} ; & K > \dfrac{\omega_0}{c} \end{cases}$$

We now add the results of (a) and (b) above to obtain:

$$\vec{A}(\vec{r},t) = \frac{\mu_0 J_{s0}\hat{z}}{2} \begin{cases} \sin\left[\omega_0 t - Kx - \sqrt{(\omega_0/c)^2-K^2}\,|y|\right] / \sqrt{(\omega_0/c)^2 - K^2} ; & K < \omega_0/c \\ \exp\left[-\sqrt{K^2-(\omega_0/c)^2}\,|y|\right] \cos(\omega_0 t - Kx) / \sqrt{K^2-(\omega_0/c)^2} ; & K > \omega_0/c \end{cases}$$

✓ $$\vec{E}(\vec{r},t) = -\vec{\nabla}\psi - \frac{\partial \vec{A}}{\partial t} = \frac{1}{2}\mu_0 J_{s0}\omega_0 \hat{z} \begin{cases} -\cos\left[\omega_0 t - Kx - \sqrt{(\omega_0/c)^2-K^2}\,|y|\right] / \sqrt{(\omega_0/c)^2-K^2} ; & K < \omega_0/c \\ \exp\left[-\sqrt{K^2-(\omega_0/c)^2}\,|y|\right] \sin(\omega_0 t - Kx) / \sqrt{K^2-(\omega_0/c)^2} ; & K > \omega_0/c \end{cases}$$

✓ $$\vec{H}(\vec{r},t) = \frac{1}{\mu_0}\vec{\nabla}\times\vec{A} = \frac{1}{\mu_0}\left(\frac{\partial A_z}{\partial y}\hat{x} - \frac{\partial A_z}{\partial x}\hat{y}\right)$$

$K < \frac{\omega_0}{c}$ → $$= -\frac{1}{2} J_{s0}\,\mathrm{Sign}(y) \cos\left[\omega_0 t - Kx - \sqrt{(\omega_0/c)^2-K^2}\,|y|\right]\hat{x} + \frac{1}{2} J_{s0} \frac{K}{\sqrt{(\omega_0/c)^2-K^2}} \cos\left[\omega_0 t - Kx - \sqrt{(\omega_0/c)^2-K^2}\,|y|\right]\hat{y}$$

$K > \frac{\omega_0}{c}$ → $$= -\frac{1}{2} J_{s0}\,\mathrm{Sign}(y) \exp\left[-\sqrt{K^2-(\omega_0/c)^2}\,|y|\right] \cos(\omega_0 t - Kx)\hat{x} - \frac{1}{2} J_{s0} \frac{K}{\sqrt{K^2-(\omega_0/c)^2}} \exp\left[-\sqrt{K^2-(\omega_0/c)^2}\,|y|\right] \sin(\omega_0 t - Kx)\hat{y}$$

Having found $\vec{E}(\vec{r},t)$ and $\vec{H}(\vec{r},t)$, it is now possible to calculate the Poynting Vector $\vec{S}(\vec{r},t)=\vec{E}(\vec{r},t)\times\vec{H}(\vec{r},t)$ and its time-average $\langle\vec{S}(\vec{r},t)\rangle$, as follows:

$$\vec{S}(\vec{r},t)=\frac{1}{4}\mu_0 J_{s0}^2\omega_0\cos^2\left[\omega_0 t-Kx-\sqrt{(\omega_0/c)^2-K^2}\,|y|\right]\frac{\mathrm{Sign}(y)\,\hat{y}+\frac{K\hat{x}}{\sqrt{(\omega_0/c)^2-K^2}}}{\sqrt{(\omega_0/c)^2-K^2}};\qquad K<\frac{\omega_0}{c}$$

$$=\frac{1}{4}Z_0 J_{s0}^2\frac{(\omega_0/c)}{(\omega_0/c)^2-K^2}\left[K\hat{x}+\mathrm{Sign}(y)\sqrt{(\omega_0/c)^2-K^2}\,\hat{y}\right]\cos^2\left[\omega_0 t-Kx-\sqrt{(\omega_0/c)^2-K^2}\,|y|\right]$$

$$\Rightarrow\ \langle\vec{S}(\vec{r},t)\rangle=\frac{1}{8}Z_0 J_{s0}^2\frac{(\omega_0/c)}{(\omega_0/c)^2-K^2}\left[K\hat{x}+\mathrm{Sign}(y)\sqrt{(\omega_0/c)^2-K^2}\,\hat{y}\right];\qquad K<\frac{\omega_0}{c}$$

Also,
$$\vec{S}(\vec{r},t)=\frac{1}{4}\mu_0 J_{s0}^2\omega_0\frac{\exp\left[-2\sqrt{K^2-(\omega_0/c)^2}\,|y|\right]}{\sqrt{K^2-(\omega_0/c)^2}}\left\{-\mathrm{Sign}(y)\sin(\omega_0 t-Kx)\cos(\omega_0 t-Kx)\,\hat{y}+\frac{K\sin^2(\omega_0 t-Kx)}{\sqrt{K^2-(\omega_0/c)^2}}\,\hat{x}\right\};\qquad K>\frac{\omega_0}{c}$$

$$=\frac{1}{4}Z_0 J_{s0}^2\frac{(\omega_0/c)}{K^2-(\omega_0/c)^2}\exp\left[-2\sqrt{K^2-(\omega_0/c)^2}\,|y|\right]\left\{K\sin^2(\omega_0 t-Kx)\,\hat{x}-\frac{1}{2}\mathrm{Sign}(y)\sqrt{K^2-(\omega_0/c)^2}\sin[2(\omega_0 t-Kx)]\,\hat{y}\right\}$$

$$\Rightarrow\ \langle\vec{S}(\vec{r},t)\rangle=\frac{1}{8}Z_0 J_{s0}^2\frac{(\omega_0/c)K}{K^2-(\omega_0/c)^2}\exp\left[-2\sqrt{K^2-(\omega_0/c)^2}\,|y|\right]\hat{x};\qquad K>\frac{\omega_0}{c}$$

✓ - In the case of $K<\omega_0/c$, the energy flows along the direction of propagation of the plane-waves on both sides of the current-carrying sheet.

✓ - In the case of $K>\omega_0/c$, the radiation on both sides of the sheet is evanescent. There is no component of $\langle\vec{S}(\vec{r},t)\rangle$ along the y-axis. There is a net flow of energy parallel to the sheet, along the x-axis, however. This energy flux is confined to a thin layer in the vicinity of the current-carrying sheet xz; note the exponential decay of the energy flux with increasing distance $|y|$ from the sheet.