Problem 17)

$$\vec{M}(\vec{r},t) = m_0 \hat{z}\,\delta(\vec{r})\,e^{\omega'' t}\left[\frac{e^{-i\omega' t} + e^{+i\omega' t}}{2}\right]$$

$$= \tfrac{1}{2} m_0 \hat{z}\,\delta(\vec{r})\left[e^{-i\omega_0 t} + e^{+i\omega_0^* t}\right].$$

Here $\omega_0 = \omega' + i\omega''$ and $\omega_0^* = \omega' - i\omega''$. At first, we'll work with the term containing $e^{-i\omega_0 t}$, then, in the end, add the corresponding formula for $e^{+i\omega_0^* t}$.

– Bound charge density $\rho_{bound}(\vec{r},t) = 0 \Rightarrow \Psi(\vec{r},t) = 0$ ✓

– Bound current density $\vec{J}_{bound}(\vec{r},t) = \frac{1}{\mu_0}\vec{\nabla}\times\vec{M}(\vec{r},t) = \frac{1}{\mu_0}\left(\frac{\partial M_z}{\partial y}\hat{x} - \frac{\partial M_z}{\partial x}\hat{y}\right)$

$$= \frac{m_0}{2\mu_0}\delta(z)\left[\delta(x)\delta'(y)\hat{x} - \delta'(x)\delta(y)\hat{y}\right]\left[e^{-i\omega_0 t} + e^{+i\omega_0^* t}\right]$$

$$\vec{J}_{bound}(\vec{k},t) = \frac{m_0}{2\mu_0}\int_{-\infty}^{\infty}\delta(z)e^{-ik_z z}dz\left\{\int_{-\infty}^{\infty}\delta(x)e^{-ik_x x}dx\int_{-\infty}^{\infty}\delta'(y)e^{-ik_y y}dy\,\hat{x} - \int_{-\infty}^{\infty}\delta'(x)e^{-ik_x x}dx\int_{-\infty}^{\infty}\delta(y)e^{-ik_y y}dy\,\hat{y}\right\}e^{-i\omega_0 t}$$

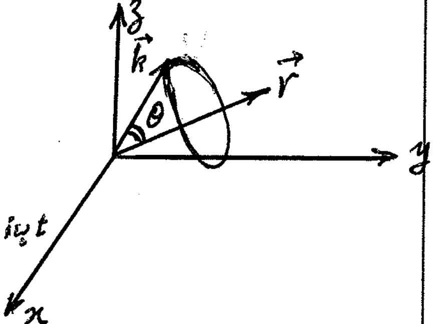
$$= \frac{m_0}{2\mu_0}\left\{-\frac{\partial}{\partial y}\left(e^{-ik_y y}\right)\Big|_{y=0}\hat{x} + \frac{\partial}{\partial x}\left(e^{-ik_x x}\right)\Big|_{x=0}\hat{y}\right\}e^{-i\omega_0 t} \leftarrow \text{Use sifting property of } \delta(\cdot)$$

$$= i\frac{m_0}{2\mu_0}\left(k_y\hat{x} - k_x\hat{y}\right)e^{-i\omega_0 t} \Rightarrow \vec{J}_{bound}(\vec{k},t) = \frac{im_0}{2\mu_0}\left(\vec{k}\times\hat{z}\right)e^{-i\omega_0 t}$$

$$\vec{A}(\vec{k},t) = \frac{\mu_0\vec{J}(\vec{k},t)}{k^2 - (\omega_0/c)^2} = \frac{im_0}{2}\,\frac{\vec{k}\times\hat{z}}{k^2 - (\omega_0/c)^2}\,e^{-i\omega_0 t}$$

$$\vec{A}(\vec{r},t) = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}\vec{A}(\vec{k},t)e^{i\vec{k}\cdot\vec{r}}d\vec{k} = \frac{m_0}{16\pi^3}\left\{\int_{-\infty}^{\infty}\frac{i\vec{k}}{k^2 - (\omega_0/c)^2}e^{i\vec{k}\cdot\vec{r}}d\vec{k}\right\}\times\hat{z}\,e^{-i\omega_0 t}$$

$$= \frac{m_0}{16\pi^3}\vec{\nabla}\left\{\int_{-\infty}^{\infty}\frac{e^{i\vec{k}\cdot\vec{r}}}{k^2 - (\omega_0/c)^2}d\vec{k}\right\}\times\hat{z}\,e^{-i\omega_0 t}$$



$$= \frac{m_0}{16\pi^3}\vec{\nabla}\left\{\int_{k=0}^{\infty}\int_{\theta=0}^{\pi}\frac{e^{ikr\cos\theta}}{k^2 - (\omega_0/c)^2}\,2\pi k^2\sin\theta\,d\theta\,dk\right\}\times\hat{z}\,e^{-i\omega_0 t}$$

$$= \frac{m_0}{8\pi^2}\vec{\nabla}\left\{\int_{k=0}^{\infty}\frac{k^2}{k^2-(\omega_0/c)^2}\int_{\theta=0}^{\pi}\sin\theta\, e^{ikr\cos\theta}\,d\theta\,dk\right\}\times\hat{z}\,e^{-i\omega_0 t}$$

$$= \frac{m_0}{4\pi^2}\vec{\nabla}\left\{\frac{1}{r}\int_0^{\infty}\frac{k\sin(kr)}{k^2-(\omega_0/c)^2}\,dk\right\}\times\hat{z}\,e^{-i\omega_0 t} = \frac{m_0}{8\pi}\vec{\nabla}\left(\frac{e^{i(\omega_0/c)r}}{r}\right)\times\hat{z}\,e^{-i\omega_0 t}$$

↖ G.R. 3.723-3 [Write $k^2-(\omega_0/c)^2$ as $k^2+(-i\omega_0/c)^2$ to make it look like the integral in G.R.]

$$= \frac{m_0}{8\pi}\frac{\partial}{\partial r}\left(\frac{e^{i\omega_0 r/c}}{r}\right)(\hat{r}\times\hat{z})\,e^{-i\omega_0 t}$$

The Corresponding Term for ω_0^* is $\frac{m_0}{8\pi}\frac{\partial}{\partial r}\left(\frac{e^{-i\omega_0^* r/c}}{r}\right)(\hat{r}\times\hat{z})\,e^{+i\omega_0^* t}$.

We now let $\omega''\to 0$, replace ω_0 with ω', add the two terms, and obtain:

$$\vec{A}(\vec{r},t) = \frac{m_0}{8\pi}\frac{\partial}{\partial r}\left\{\frac{e^{i\omega'(t-r/c)}+e^{-i\omega'(t-r/c)}}{r}\right\}(\hat{r}\times\hat{z})$$

$$= \frac{m_0}{4\pi}\frac{\partial}{\partial r}\left\{\frac{\cos[\omega'(t-r/c)]}{r}\right\}(-\sin\theta)\hat{\phi} \Rightarrow$$

$$\vec{A}(\vec{r},t) = \frac{m_0\sin\theta}{4\pi r}\left\{\frac{1}{r}\cos[\omega'(t-r/c)] - \left(\frac{\omega'}{c}\right)\sin[\omega'(t-r/c)]\right\}\hat{\phi}$$

$$\vec{E}(\vec{r},t) = -\vec{\nabla}\psi - \frac{\partial\vec{A}}{\partial t} = \frac{m_0\omega'\sin\theta}{4\pi r}\left\{\frac{1}{r}\sin[\omega'(t-r/c)] + \left(\frac{\omega'}{c}\right)\cos[\omega'(t-r/c)]\right\}\hat{\phi}$$

$$\vec{H}(\vec{r},t) = \frac{1}{\mu_0}\vec{\nabla}\times\vec{A} = \frac{1}{\mu_0}\left\{\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\phi)\hat{r} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\hat{\theta}\right\}$$

$$= \frac{m_0}{4\pi\mu_0}\left\{\frac{2\cos\theta}{r^2}\left[\frac{1}{r}\cos(\omega'\ldots) - \left(\frac{\omega'}{c}\right)\sin(\omega'\ldots)\right]\hat{r} - \frac{\sin\theta}{r}\frac{\partial}{\partial r}\left[\frac{1}{r}\cos(\omega'\ldots) - \left(\frac{\omega'}{c}\right)\sin(\omega'\ldots)\right]\hat{\theta}\right\}$$

$$= \frac{m_0}{4\pi\mu_0}\left\{\frac{2\cos\theta}{r^2}\left[\frac{1}{r}\cos(\omega'\ldots) - \left(\frac{\omega'}{c}\right)\sin(\omega'\ldots)\right]\hat{r} - \frac{\sin\theta}{r}\left[-\frac{1}{r^2}\cos(\omega'\ldots) + \frac{1}{r}\left(\frac{\omega'}{c}\right)\sin(\omega'\ldots) + \left(\frac{\omega'}{c}\right)^2\cos(\omega'\ldots)\right]\hat{\theta}\right\}$$

$$\vec{H}(\vec{r},t) = -\frac{m_0\sin\theta\,\hat{\theta}}{4\pi\mu_0 r}\left(\frac{\omega'}{c}\right)^2\cos[\omega'(t-r/c)] - \frac{m_0}{4\pi\mu_0 r^2}\left\{\left(\frac{\omega'}{c}\right)\sin[\omega'(t-r/c)] - \frac{1}{r}\cos[\omega'(t-r/c)]\right\}(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta})$$