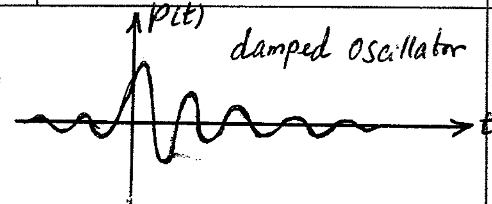
Problem 16)

$$\vec{p}(\vec{r},t) = \frac{p_0 \hat{z}\, \delta(\vec{r})\, e^{-\alpha t} \sin(\omega_0 t+\phi_0)}{1+e^{-\beta t}}.$$



Fourier Transforming the time-dependent part:

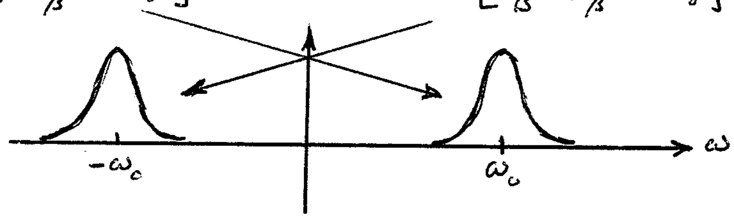
$$\int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt = \frac{1}{2i}\int_{-\infty}^{\infty} \frac{e^{-\alpha t}\left[e^{i(\omega_0 t+\phi_0)} - e^{-i(\omega_0 t+\phi_0)}\right] e^{-i\omega t}}{1+e^{-\beta t}}\, dt$$

$$= \frac{1}{2i}\int_{-\infty}^{\infty} \frac{e^{-(\alpha - i\omega_0 + i\omega)t}\, e^{i\phi_0} - e^{-(\alpha+i\omega_0+i\omega)t}\, e^{-i\phi_0}}{1+e^{-\beta t}}\, dt$$

$$= \frac{1}{2i\beta}\left\{ e^{i\phi_0}\int_{-\infty}^{\infty} \frac{e^{-(\alpha - i\omega_0+i\omega)x/\beta}}{1+e^{-x}}\, dx - e^{-i\phi_0}\int_{-\infty}^{\infty} \frac{e^{-(\alpha+i\omega_0+i\omega)x/\beta}}{1+e^{-x}}\, dx \right\}$$

G.R. 3.311-9

$$= \frac{\pi e^{i\phi_0}}{2i\beta}\, \frac{1}{\sin\left[\frac{\pi\alpha}{\beta} + i\frac{\pi}{\beta}(\omega-\omega_0)\right]} - \frac{\pi e^{-i\phi_0}}{2i\beta}\, \frac{1}{\sin\left[\frac{\pi\alpha}{\beta} + i\frac{\pi}{\beta}(\omega+\omega_0)\right]}$$



The two sides of the frequency spectrum are generally narrow and far apart. Therefore, we can ignore their overlap and focus only on one side:

Power Spectrum of the right-hand-side $= \left| \dfrac{\pi e^{i\phi_0}}{2i\beta \sin\left[\frac{\pi\alpha}{\beta} + i\frac{\pi}{\beta}(\omega-\omega_0)\right]} \right|^2$

$$= \frac{\pi^2}{4\beta^2} \left| \sin\left(\frac{\pi\alpha}{\beta}\right) \cosh\left(\frac{\pi}{\beta}(\omega-\omega_0)\right) + i \cos\left(\frac{\pi\alpha}{\beta}\right) \sinh\left(\frac{\pi}{\beta}(\omega-\omega_0)\right) \right|^{-2}$$ ← G.R. 1.313-3

$$= \frac{\pi^2}{4\beta^2}\left[\sin^2\left(\frac{\pi\alpha}{\beta}\right) \cosh^2\left(\frac{\pi}{\beta}(\omega-\omega_0)\right) + \cos^2\left(\frac{\pi\alpha}{\beta}\right) \sinh^2\left(\frac{\pi}{\beta}(\omega-\omega_0)\right) \right]^{-1}$$

$\hookrightarrow 1+\sinh^2(\cdots)$

$$= \frac{\pi^2}{4\beta^2}\left[\sin^2\left(\frac{\pi\alpha}{\beta}\right) + \sinh^2\left(\frac{\pi}{\beta}(\omega-\omega_0)\right) \right]^{-1} = \frac{\pi^2}{4\beta^2}\left[\frac{1}{2} - \frac{1}{2}\cos\left(\frac{2\pi\alpha}{\beta}\right) + \frac{1}{2}\cosh\left(\frac{2\pi}{\beta}(\omega-\omega_0)\right) - \frac{1}{2} \right]^{-1}$$

↖ G.R. 1.322-1

$$= \frac{\pi^2}{2\beta^2}\left\{ \cosh\left[\frac{2\pi}{\beta}(\omega-\omega_0)\right] - \cos\left(\frac{2\pi\alpha}{\beta}\right) \right\}^{-1}$$

For an estimate of the spectral line-width, we expand the Cosh and Cos functions in a Taylor series, and use the "small argument" approximation as follows:

$$\cosh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\ ; \qquad \cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

$$\text{Spectral line-shape} \simeq \frac{\left(\frac{\pi^2}{2\beta^2}\right)}{\cancel{1}+\frac{1}{2}\left[\frac{2\pi}{\beta}(\omega-\omega_c)\right]^2-\cancel{1}+\frac{1}{2}\left(\frac{2\pi\alpha}{\beta}\right)^2} = \frac{1/4}{(\omega-\omega_c)^2+\alpha^2} \Rightarrow \text{line width} \simeq \alpha$$

$$\text{Total radiated power} = \frac{\mu_0|P_0|^2}{12\pi c}\left(\frac{\pi^2}{2\beta^2}\right)\int_{\omega=0}^{\infty}\frac{\omega^4}{\cosh\left[\frac{2\pi}{\beta}(\omega-\omega_c)\right]-\cos\left(\frac{2\pi\alpha}{\beta}\right)}d\omega$$

$$= \frac{\mu_0|P_0|^2}{48c\beta}\int_{-\infty}^{\infty}\frac{\left(\omega_0+\frac{\beta}{2\pi}x\right)^4}{\cosh(x)-\cos\left(\frac{2\pi\alpha}{\beta}\right)}dx = \frac{\mu_0|P_0|^2}{24c\beta}\int_0^{\infty}\frac{\omega_0^4+6(\beta/2\pi)^2\omega_0^2x^2+(\beta/2\pi)^4x^4}{\cosh(x)-\cos(2\pi\alpha/\beta)}dx$$

$$= \frac{\mu_0|P_0|^2}{24c\beta}\left\{\omega_0^4\int_0^{\infty}\frac{dx}{\cosh(x)+\cos\left(\pi-\frac{2\pi\alpha}{\beta}\right)}+6\left(\frac{\beta\omega_0}{2\pi}\right)^2\int_0^{\infty}\frac{x^2dx}{\cosh(x)+\cos\left(\pi-\frac{2\pi\alpha}{\beta}\right)}+\left(\frac{\beta}{2\pi}\right)^4\int_0^{\infty}\frac{x^4dx}{\cosh(x)+\cos\left(\pi-\frac{2\pi\alpha}{\beta}\right)}\right\}$$

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$$= \frac{\mu_0|P_0|^2}{24c\beta}\left\{\frac{\left(\pi-\frac{2\pi\alpha}{\beta}\right)\omega_0^4}{\sin(2\pi\alpha/\beta)}+6\left(\frac{\beta\omega_0}{2\pi}\right)^2\left(\frac{\pi-\frac{2\pi\alpha}{\beta}}{3}\right)\frac{\pi^2-(\pi-2\pi\alpha/\beta)^2}{\sin(2\pi\alpha/\beta)}+\left(\frac{\beta}{2\pi}\right)^4\left(\frac{\pi-2\pi\alpha/\beta}{15}\right)\times\frac{[\pi^2-(\pi-2\pi\alpha/\beta)^2][7\pi^2-3(\pi-2\pi\alpha/\beta)^2]}{\sin(2\pi\alpha/\beta)}\right\}$$

$$= \frac{\pi\mu_0\left(1-\frac{2\alpha}{\beta}\right)|P_0|^2}{24c\beta\sin(2\pi\alpha/\beta)}\left\{\omega_0^4+\frac{1}{2}\beta^2\omega_0^2\left(\frac{4\alpha}{\beta}-\frac{4\alpha^2}{\beta^2}\right)+\frac{\beta^4}{16\times15}\left(\frac{4\alpha}{\beta}-\frac{4\alpha^2}{\beta^2}\right)\left(4+\frac{12\alpha}{\beta}-\frac{12\alpha^2}{\beta^2}\right)\right\}$$

$$= \frac{\pi\mu_0\left(1-\frac{2\alpha}{\beta}\right)|P_0|^2}{24c\beta\sin(2\pi\alpha/\beta)}\left\{\omega_0^4+2\alpha(\beta-\alpha)\omega_0^2+\frac{1}{15}\alpha(\beta-\alpha)(\beta^2+3\alpha\beta-3\alpha^2)\right\}$$

$$\simeq \frac{\mu_0\left(1-\frac{2\alpha}{\beta}\right)|P_0|^2}{48c\alpha}\left[\omega_0^4+2\alpha(\beta-\alpha)\omega_0^2+\frac{1}{5}\alpha^2(\beta-\alpha)^2+\frac{1}{15}\alpha\beta^2(\beta-\alpha)\right]$$

← approximating $\sin(2\pi\alpha/\beta)$ with $2\pi\alpha/\beta$

Note that the area under the line-shape is:

$$\text{Area} = \frac{\pi^2}{2\beta^2}\int_0^{\infty}\frac{d\omega}{\cosh\left[\frac{2\pi}{\beta}(\omega-\omega_c)\right]-\cos(2\pi\alpha/\beta)} = \frac{\pi}{4\beta}\int_{-\infty}^{\infty}\frac{dx}{\cosh(x)+\cos(\pi-2\pi\alpha/\beta)}$$

$$= \frac{\pi}{2\beta}\,\frac{(\pi-2\pi\alpha/\beta)}{\sin(2\pi\alpha/\beta)} = \frac{\pi^2\left(1-\frac{2\alpha}{\beta}\right)}{2\beta\sin(2\pi\alpha/\beta)} \simeq \frac{\pi}{4\alpha}\left(1-\frac{2\alpha}{\beta}\right)$$
