Problem 15)

$$\vec{P}(\vec{r},t) = P_o\hat{z}\,\delta(\vec{r})\,e^{\omega' t}\cos(\omega'' t) = \tfrac{1}{2}P_o\hat{z}\,\delta(\vec{r})\{e^{\omega t} + e^{\omega^* t}\}$$

$\omega = \omega' + i\omega''$

The potentials and the fields can be found using the same method as used in Problem 16 with $-i\omega$ replaced by ω (or ω^*). We'll have:

$$\psi(\vec{k},\omega) = -\frac{i\vec{k}\cdot\vec{P}(\vec{k},\omega)}{\epsilon_o(k^2+\omega^2/c^2)} = -\frac{iP_o}{2\epsilon_o}\,\frac{k_z}{k^2+\omega^2/c^2}$$

$$\vec{A}(\vec{k},\omega) = \frac{\mu_o\omega\vec{P}(\vec{k},\omega)}{k^2+\omega^2/c^2} = \frac{P_o\hat{z}}{2\epsilon_o}\,\frac{\omega/c^2}{k^2+\omega^2/c^2}$$

$$\vec{E}(\vec{k},\omega) = -i\vec{k}\psi(\vec{k},\omega) - \omega\vec{A}(\vec{k},\omega) = \underbrace{-\frac{P_o}{2\epsilon_o}\,\frac{k_z\vec{k}}{k^2+\omega^2/c^2}}\; - \frac{P_o\hat{z}}{2\epsilon_o}\,\frac{\omega^2/c^2}{k^2+\omega^2/c^2} = -\frac{P_o}{2\epsilon_o}\,\frac{k_z\vec{k}+(\omega^2/c^2)\hat{z}}{k^2+\omega^2/c^2}$$

Time-rate-of-change of energy of $\vec{P}(\vec{r},t)$: $\quad \frac{\partial}{\partial t}\mathcal{E}(\vec{r},t) = \vec{E}(\vec{r},t)\cdot\frac{\partial}{\partial t}\vec{P}(\vec{r},t) \Rightarrow$

$$\frac{\partial}{\partial t}\mathcal{E}(\vec{r},t) = -\frac{P_o}{2\epsilon_o}\frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}\left\{\frac{k_z\vec{k}+(\omega^2/c^2)\hat{z}}{k^2+\omega^2/c^2}e^{\omega t} + \frac{k_z\vec{k}+(\omega^{*2}/c^2)\hat{z}}{k^2+\omega^{*2}/c^2}e^{\omega^* t}\right\}e^{i\vec{k}\cdot\vec{r}}\,d\vec{k}$$

$$\cdot\,\frac{P_o\hat{z}}{2}\,\delta(\vec{r})\,(\omega e^{\omega t} + \omega^* e^{\omega^* t}) \Rightarrow$$

$$\int_{-\infty}^{\infty}\frac{\partial}{\partial t}\mathcal{E}(\vec{r},t)\,d\vec{r} = \frac{\partial}{\partial t}\int_{-\infty}^{\infty}\mathcal{E}(\vec{r},t)\,d\vec{r} = -\frac{P_o^2}{4(2\pi)^3\epsilon_o}\int_{-\infty}^{\infty}\left[\frac{k_z^2+\omega^2/c^2}{k^2+\omega^2/c^2}e^{\omega t} + \frac{k_z^2+(\omega^*/c)^2}{k^2+(\omega^*/c)^2}e^{\omega^* t}\right](\omega e^{\omega t}+\omega^* e^{\omega^* t})$$

$$\times\Big(\underbrace{\int_{-\infty}^{\infty} e^{i\vec{k}\cdot\vec{r}}\,\delta(\vec{r})\,d\vec{r}}_{1}\Big)\,d\vec{k}$$

$$= -\frac{P_o^2}{4\epsilon_o(2\pi)^3}(\omega e^{\omega t}+\omega^* e^{\omega^* t})\int_{k=0}^{\infty}\int_{\theta=0}^{\pi}\left[\frac{k^2\cos^2\theta+(\omega/c)^2}{k^2+\omega^2/c^2}e^{\omega t} + \frac{k^2\cos^2\theta+(\omega^*/c)^2}{k^2+(\omega^*/c)^2}e^{\omega^* t}\right]2\pi k^2\sin\theta\,dk\,d\theta$$

$$= -\frac{P_o^2}{4\epsilon_o(2\pi)^2}(\omega e^{\omega t}+\omega^* e^{\omega^* t})\int_{k=0}^{\infty}\left[\frac{\frac{2}{3}k^4+2(\omega/c)^2k^2}{k^2+(\omega/c)^2}e^{\omega t} + \frac{\frac{2}{3}k^4+2(\omega^*/c)^2k^2}{k^2+(\omega^*/c)^2}e^{\omega^* t}\right]dk$$

$$= -\frac{P_o^2}{24\pi^2\epsilon_o}(\omega e^{\omega t}+\omega^* e^{\omega^* t})\int_{k=0}^{\infty}\left[\frac{(k^2+\frac{\omega^2}{c^2})(k^2+\frac{2\omega^2}{c^2})-2(\omega/c)^4}{k^2+\omega^2/c^2}e^{\omega t} + \frac{(k^2+\frac{\omega^{*2}}{c^2})(k^2+\frac{2\omega^{*2}}{c^2})-2(\frac{\omega^*}{c})^4}{k^2+(\omega^*/c)^2}e^{\omega^* t}\right]dk$$

$$= -\frac{P_o^2}{12\pi^2\epsilon_o}\left\{\mathrm{Re}\left[\omega e^{2\omega t}\int_0^{\infty}\left(k^2+\frac{2\omega^2}{c^2}-\frac{2(\omega/c)^4}{k^2+\omega^2/c^2}\right)dk\right] + \mathrm{Re}\left[e^{(\omega+\omega^*)t}\int_0^{\infty}\omega^*\left(k^2+\frac{2\omega^2}{c^2}-\frac{2(\omega/c)^4}{k^2+\omega^2/c^2}\right)dk\right]\right\}$$

Now, the first term in the above expression oscillates with a frequency of 2ω, thus averaging out to zero. The remaining term may be further simplified as follows:

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty}\mathcal{E}(\vec{r},t)\,d\vec{r} = -\frac{P_0^2 e^{2\omega' t}}{12\pi^2\epsilon_0}\,\mathrm{Re}\left\{\int_0^\infty(\omega'-i\omega'')\left(k^2+\frac{2\omega'^2}{c^2}-\frac{2\omega''^2}{c^2}+\frac{4i\omega'\omega''}{c^2}\right)dk-\int_0^\infty\frac{2\omega^*(\omega/c)^4}{k^2+\omega^2/c^2}dk\right\}$$

$$= -\frac{P_0^2 e^{2\omega' t}}{12\pi^2\epsilon_0}\left\{\omega'\int_0^\infty\left(k^2+\frac{2\omega'^2}{c^2}+\frac{2\omega''^2}{c^2}\right)dk-\mathrm{Re}\left[\omega^*(\omega/c)^4\int_{-\infty}^{\infty}\frac{dk}{k^2+\omega^2/c^2}\right]\right\}$$

Again, we ignore the first term in the limit $\omega'\to 0$, even though the integral is infinite. The integral in the second term may be evaluated as follows:

$$\int_{-\infty}^{\infty}\frac{dk}{k^2+\omega^2/c^2} = \int_{-\infty}^{\infty}\frac{dk}{(k-i\omega/c)(-k-i\omega/c)} = -\frac{i\pi}{-i\omega/c} = \frac{\pi}{\omega/c}$$

G.R. 3-112-2

root in the upper-half plane

$$\Rightarrow -\mathrm{Re}\left[\omega^*(\omega/c)^4\int_{-\infty}^{\infty}\frac{dk}{k^2+\omega^2/c^2}\right] = -\pi\,\mathrm{Re}\left[\omega^*(\omega/c)^3\right] = -\frac{\pi}{c^3}\,\mathrm{Re}\left(|\omega|^2\omega^2\right)$$

$$= -\frac{\pi\mu_0\epsilon_0}{c}(\omega'^2+\omega''^2)(\omega'^2-\omega''^2) = \frac{\pi\mu_0\epsilon_0}{c}(\omega''^4-\omega'^4)$$

Therefore, $\displaystyle\frac{\partial}{\partial t}\int_{-\infty}^{\infty}\mathcal{E}(\vec{r},t)\,d\vec{r} = -\frac{\mu_0 P_0^2(\omega''^4-\omega'^4)}{12\pi c}e^{2\omega' t} \xrightarrow[\omega'\to 0]{} -\frac{\mu_0 P_0^2\omega''^4}{12\pi c}$

The minus sign indicates that the energy comes out of the oscillating dipole (rather than going into it). This is the same energy that we found in Problem 17 by direct calculation using the Poynting vector of the radiated field.