Problem 14)

Point-dipole oscillator $\vec{P}(\vec{r},t) = P_0\hat{z}\,\delta(\vec{r})e^{\omega'' t}\cos(\omega' t) \Rightarrow$

$$\vec{P}(\vec{r},t) = \tfrac{1}{2}P_0\hat{z}\,\delta(\vec{r})e^{\omega'' t}(e^{i\omega' t}+e^{-i\omega' t}) = \tfrac{1}{2}P_0\hat{z}\,\delta(\vec{r})e^{-i(\omega'+i\omega'')t} + c.c.$$

Defining $\omega = \omega' + i\omega''$, where $\omega'' \geq 0$, we will find the scalar and vector potentials for $\vec{P}(\vec{r},t) = P_0\hat{z}\,\delta(\vec{r})e^{-i\omega t}$, whose spatial Fourier transform is given by $\vec{P}(\vec{k},t) = P_0\hat{z}e^{-i\omega t}$. The spatial parts of $\psi(\vec{k},t)$ and $\vec{A}(\vec{k},t)$ are thus given by

$$\psi(\vec{k}) = -\frac{iP_0}{\epsilon_0}\,\frac{\vec{k}\cdot\hat{z}}{k^2-\omega^2/c^2}\;;\quad \vec{A}(\vec{k}) = -i\omega\mu_0 P_0\,\frac{\hat{z}}{k^2-\omega^2/c^2}\;;\quad \text{(See problem 16).}$$

Inverse Fourier Transformation: $\psi(\vec{r}) = -\frac{iP_0}{\epsilon_0}\frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}\frac{k_z}{k^2-\omega^2/c^2}e^{+i\vec{k}\cdot\vec{r}}\,d\vec{k}$

$$= -\frac{P_0}{\epsilon_0}\frac{1}{(2\pi)^3}\frac{\partial}{\partial z}\int_{-\infty}^{\infty}\frac{e^{i\vec{k}\cdot\vec{r}}}{k^2-\omega^2/c^2}\,d\vec{k} = -\frac{P_0}{(2\pi)^3\epsilon_0}\frac{\partial}{\partial z}\int_{k=0}^{\infty}\int_{\theta=0}^{\pi}\frac{e^{ikr\cos\theta}}{k^2-\omega^2/c^2}\,2\pi k^2\sin\theta\,dk\,d\theta$$

$$= \frac{-iP_0}{4\pi^2\epsilon_0}\frac{\partial}{\partial z}\left\{\frac{1}{r}\int_{k=0}^{\infty}\frac{k}{k^2-\omega^2/c^2}e^{ikr\cos\theta}\Big|_{\theta=0}^{\pi}dk\right\} = -\frac{2P_0}{4\pi^2\epsilon_0}\frac{\partial}{\partial z}\left\{\frac{1}{r}\int_{k=0}^{\infty}\frac{k\sin(kr)}{k^2-\omega^2/c^2}\,dk\right\}$$

$$= -\frac{P_0}{2\pi^2\epsilon_0}\frac{\partial}{\partial z}\left\{\frac{1}{r}\int_0^{\infty}\frac{k\sin(kr)}{k^2+(-i\omega/c)^2}\,dk\right\} = -\frac{P_0}{4\pi\epsilon_0}\frac{\partial}{\partial z}\left(\frac{e^{i\omega r/c}}{r}\right)$$

G.R. 3.723-3

Therefore, $\psi(\vec{r},t) = -\frac{P_0}{4\pi\epsilon_0}\frac{\partial}{\partial z}\left[\frac{e^{-i\omega t}e^{i\omega r/c}+e^{+i\omega t}e^{-i\omega r/c}}{2r}\right] = -\frac{P_0}{4\pi\epsilon_0}\frac{\partial}{\partial z}\left\{\frac{1}{r}\cos[\omega(t-\frac{r}{c})]\right\}$

$$= -\frac{P_0}{4\pi\epsilon_0}\left\{-\frac{1}{r^2}\left(\frac{\partial r}{\partial z}\right)\cos[\omega(t-r/c)] + \frac{1}{r}\left(\frac{\omega}{c}\right)\left(\frac{\partial r}{\partial z}\right)\sin[\omega(t-r/c)]\right\}$$

$$r = \sqrt{x^2+y^2+z^2}\qquad \frac{\partial r}{\partial z} = \frac{z}{r} = \cos\theta$$

$$= \frac{P_0}{4\pi\epsilon_0}\left\{\frac{z}{r^3}\cos[\omega(t-r/c)] - \left(\frac{\omega}{c}\right)\left(\frac{z}{r^2}\right)\sin[\omega(t-r/c)]\right\} \Rightarrow$$

$$\psi(\vec{r},t) = \frac{P_0\cos\theta}{4\pi\epsilon_0 r}\left\{\frac{1}{r}\cos[\omega(t-r/c)] - \left(\frac{\omega}{c}\right)\sin[\omega(t-r/c)]\right\}$$

Vector potential: Inverse Fourier Transform: $\vec{A}(\vec{r}) = -\frac{i\omega\mu_0 P_0\hat{z}}{(2\pi)^3}\int_{-\infty}^{\infty}\frac{e^{+i\vec{k}\cdot\vec{r}}}{k^2-\omega^2/c^2}\,d\vec{k}$

$$= -\frac{i\omega\mu_0 P_0\hat{z}}{(2\pi)^3}\int_{k=0}^{\infty}\int_{\theta=0}^{\pi}\frac{e^{ikr\cos\theta}}{k^2-\omega^2/c^2}\,2\pi k^2\sin\theta\,dk\,d\theta = \frac{\omega\mu_0 P_0\hat{z}}{4\pi^2 r}\int_{k=0}^{\infty}\frac{k}{k^2-\omega^2/c^2}e^{ikr\cos\theta}\Big|_{\theta=0}^{\pi}dk$$

$$= -\frac{i\omega\mu_0 P_0\hat{z}}{2\pi^2 r}\int_{k=0}^{\infty}\frac{k\sin(kr)}{k^2-\omega^2/c^2}\,dk = -\frac{i\omega\mu_0 P_0\hat{z}}{4\pi r}e^{i\omega r/c}$$

Therefore, $\vec{A}(\vec{r},t) = \frac{\mu_0 p_0 \hat{z}}{4\pi r}\frac{1}{2}\left\{-i\omega e^{i\omega r/c} e^{-i\omega t} + i\omega e^{-i\omega r/c} e^{+i\omega t}\right\} \Rightarrow$

$$\vec{A}(\vec{r},t) = -\frac{\mu_0 p_0 \omega \hat{z}}{4\pi r}\sin[\omega(t - r/c)]$$

$$\vec{B}(\vec{r},t) = \vec{\nabla}\times\vec{A}(\vec{r},t) = \vec{\nabla}\times\left\{-\frac{\mu_0 p_0 \omega}{4\pi r}\sin[\omega(t-r/c)]\,(\hat{r}\cos\theta - \hat{\theta}\sin\theta)\right\}$$

$$= -\frac{\mu_0 p_0 \omega}{4\pi}\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(-\sin\omega(t-r/c)\sin\theta\right) - \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{r}\sin\omega(t-r/c)\cos\theta\right)\right\}\hat{\phi}$$

$$= -\frac{\mu_0 p_0 \omega}{4\pi}\left\{\frac{1}{r}\left(\frac{\omega}{c}\right)\cos[\omega(t-r/c)]\sin\theta + \frac{1}{r^2}\sin[\omega(t-r/c)]\sin\theta\right\}\hat{\phi} \Rightarrow$$

$$\vec{H}(\vec{r},t) = -\frac{p_0\omega\sin\theta\,\hat{\phi}}{4\pi r}\left\{\left(\frac{\omega}{c}\right)\cos[\omega(t-r/c)] + \frac{1}{r}\sin[\omega(t-r/c)]\right\}$$

$$\vec{E}(\vec{r},t) = -\vec{\nabla}\psi - \frac{\partial\vec{A}}{\partial t} = -\frac{\partial\psi}{\partial r}\hat{r} - \frac{1}{r}\frac{\partial\psi}{\partial\theta}\hat{\theta} + \frac{\mu_0 p_0\omega^2}{4\pi r}\cos[\omega(t-r/c)]\,(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta})$$

$$= -\frac{p_0\cos\theta\,\hat{r}}{4\pi\epsilon_0}\left\{\frac{-2}{r^3}\cos[\omega(t-r/c)] + \frac{1}{r^2}\left(\frac{\omega}{c}\right)\sin[\omega(t-r/c)] + \frac{1}{r^2}\sin[\omega(t-r/c)]\left(\frac{\omega}{c}\right)\right.$$

$$\left. + \frac{1}{r}\left(\frac{\omega}{c}\right)^2\cos[\omega(t-r/c)]\right\} + \frac{p_0\sin\theta\,\hat{\theta}}{4\pi\epsilon_0 r^2}\left\{\frac{1}{r}\cos[\omega(t-r/c)] - \left(\frac{\omega}{c}\right)\sin[\omega(t-r/c)]\right\}$$

$$+ \frac{p_0}{4\pi\epsilon_0 r}\left(\frac{\omega}{c}\right)^2(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta})\cos[\omega(t-r/c)] \Rightarrow$$

$$\vec{E}(\vec{r},t) = \frac{p_0\cos\theta\,\hat{r}}{4\pi\epsilon_0 r}\left\{\frac{2}{r^2}\cos[\omega(t-r/c)] - \frac{2}{r}\left(\frac{\omega}{c}\right)\sin[\omega(t-r/c)]\right\}$$

$$+ \frac{p_0\sin\theta\,\hat{\theta}}{4\pi\epsilon_0 r}\left\{\left(\frac{1}{r^2} - \frac{\omega^2}{c^2}\right)\cos[\omega(t-r/c)] - \frac{1}{r}\left(\frac{\omega}{c}\right)\sin[\omega(t-r/c)]\right\} \Rightarrow$$

$$\vec{E}(\vec{r},t) = -\frac{p_0\sin\theta\,\hat{\theta}}{4\pi\epsilon_0 r}\left(\frac{\omega}{c}\right)^2\cos[\omega(t-r/c)] - \frac{p_0}{4\pi\epsilon_0 r^2}\left\{\left(\frac{\omega}{c}\right)\sin[\omega(t-r/c)] - \frac{1}{r}\cos[\omega(t-r/c)]\right\}$$

$$\times(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta})$$

In the far field the terms that decrease as $1/r^2$ and $1/r^3$ may be ignored. The Poynting vector (in the far-field) is thus given by:

$$\vec{S}(\vec{r},t) = \vec{E}\times\vec{H} = \frac{p_0\sin\theta\,\hat{\theta}}{4\pi\epsilon_0 r}\left(\frac{\omega}{c}\right)^2\cos[\omega(t-r/c)] \times \frac{p_0\omega\sin\theta\,\hat{\phi}}{4\pi r}\left(\frac{\omega}{c}\right)\cos[\omega(t-r/c)]$$

$$= \frac{p_0^2\sin^2\theta\,\omega^4}{16\pi^2\epsilon_0 c^3 r^2}\cos^2[\omega(t-r/c)]\,\hat{r} \Longrightarrow \langle\vec{S}(\vec{r},t)\rangle = \frac{\mu_0 p_0^2\sin^2\theta\,\omega^4}{32\pi^2 c r^2}\hat{r}$$

The total radiated power is obtained by integrating over the surface of an arbitrary sphere of radius r. We find:

$$\text{Total radiated power} = \int_{\theta=0}^{\pi} 2\pi r^2 \sin\theta \langle \vec{S}(\vec{r},t)\rangle \, d\theta$$

$$= \frac{\mu_o P_o^2 \omega^4}{16\pi c} \int_0^{\pi} \sin^3\theta \, d\theta = \frac{\mu_o P_o^2 \omega^4}{12\pi c}$$

Digression: The scalar potential $\psi(\vec{r})$ may also be obtained by direct integration, as follows:

$$\int_{-\infty}^{\infty} \frac{k_z}{k^2-\omega^2/c^2} e^{i\vec{k}\cdot\vec{r}} d\vec{k} = \int_{k=0}^{\infty}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \frac{k\cos\theta}{k^2-\omega^2/c^2} e^{ik(\sin\theta\cos\phi\, x + \sin\theta\sin\phi\, y + \cos\theta\, z)} k^2 \sin\theta \, dk\, d\theta\, d\phi$$

$$= \int_{k=0}^{\infty}\int_{\theta=0}^{\pi} \frac{k^3 \sin\theta\cos\theta\, e^{ikz\cos\theta}}{k^2-\omega^2/c^2} \left\{ \int_{\phi=0}^{2\pi} e^{ik\sin\theta(x\cos\phi + y\sin\phi)} d\phi \right\} dk\, d\theta$$

← G.R. 3.937-2

$$= 2\pi \int_{k=0}^{\infty} \frac{k^3}{k^2-\omega^2/c^2} \left\{ \int_{\theta=0}^{\pi} e^{ikz\cos\theta} J_0(k\sqrt{x^2+y^2}\sin\theta) \sin\theta\cos\theta\, d\theta \right\} dk$$

$$= 2\pi \int_{k=0}^{\infty} \frac{k^3}{k^2-\omega^2/c^2} \left\{ 2i \int_0^1 \sin(kz\sqrt{1-u^2}) J_0(k\sqrt{x^2+y^2}\, u)\, u\, du \right\} dk$$

$$= \frac{4\pi i}{x^2+y^2} \int_{k=0}^{\infty} \frac{k}{k^2-\omega^2/c^2} \left\{ \int_0^{k\sqrt{x^2+y^2}} \sin\left(kz\sqrt{1-\frac{v^2}{k^2(x^2+y^2)}}\right) J_0(v)\, v\, dv \right\} dk$$

← G.R. 6.738-1

$$= \frac{4\pi i}{x^2+y^2} \int_{k=0}^{\infty} \frac{k}{k^2-\omega^2/c^2} \left\{ \sqrt{\frac{\pi}{2}} (k\sqrt{x^2+y^2})^{3/2} \left(\frac{z}{\sqrt{x^2+y^2}}\right)\left(1+\frac{z^2}{x^2+y^2}\right)^{-3/4} J_{3/2}(k\sqrt{x^2+y^2+z^2}) \right\} dk$$

$$= 4\pi i \left(\frac{z}{r^{3/2}}\right)\sqrt{\frac{\pi}{2}} \int_0^{\infty} \frac{k^{5/2}}{k^2-\omega^2/c^2} J_{3/2}(kr)\, dk; \quad \leftarrow J_{3/2}(r) = \sqrt{\frac{2}{\pi r}}\left(\frac{\sin r}{r} - \cos r\right); \text{ G.R. 8.464-3}$$

← G.R. 6.565-4 [write $k^2-\omega^2/c^2 = k^2+(-i\omega/c)^2$; $Re(-i\omega/c)=\omega''/c>0$]

$$= 4\pi i \left(\frac{z}{r^{3/2}}\right)\sqrt{\frac{\pi}{2}} (-i\omega/c)^{3/2} K_{3/2}\left(-i\frac{\omega}{c}r\right); \quad \leftarrow K_{3/2}(r) = \sqrt{\frac{\pi}{2r}}\, e^{-r}\left(1+\frac{1}{r}\right); \text{ G.R. 8.468}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{k_3}{k^2-\omega^2/c^2} e^{i\vec{k}\cdot\vec{r}} d\vec{k} = 4\pi i \left(\frac{\partial}{r^{3/2}}\right)\sqrt{\pi/2}\left(-i\frac{\omega}{c}\right)^{3/2}\sqrt{\frac{\pi}{-2i\frac{\omega}{c}r}}\, e^{i\frac{\omega}{c}r}\left(1+\frac{1}{-i\omega/c\, r}\right)$$

$$= 2\pi^2\left(\frac{\omega}{c}\right)\left(\frac{\partial}{r^2}\right)\left(1+\frac{i}{(\omega/c)r}\right)e^{i(\omega/c)r}$$

Having completed the integration, we may now assume that ω is real-valued (i.e., set $\omega''=0$). We'll have:

$$\psi(\vec{r},t) = -\frac{ip_0}{2\epsilon_0}\frac{1}{(2\pi)^3}\left\{2\pi^2\left(\frac{\omega}{c}\right)\left(\frac{\partial}{r^2}\right)\left(1+\frac{i}{(\omega/c)r}\right)e^{i(\omega/c)r}e^{-i\omega t} + 2\pi^2\left(-\frac{\omega}{c}\right)\left(\frac{\partial}{r^2}\right)\left(1-\frac{i}{(\omega/c)r}\right)e^{-i(\frac{\omega}{c})r}e^{+i\omega t}\right\}$$

$$= -i\frac{p_0}{8\pi\epsilon_0}\left(\frac{\omega}{c}\right)\left(\frac{\partial}{r^2}\right)\left\{\left(1+\frac{i}{(\omega/c)r}\right)e^{-i\omega(t-r/c)} - \left(1-\frac{i}{(\omega/c)r}\right)e^{+i\omega(t-r/c)}\right\}$$

∂/r

$$= \frac{p_0\cos\theta}{4\pi\epsilon_0 r}\left\{\frac{1}{r}\cos[\omega(t-r/c)] - \left(\frac{\omega}{c}\right)\sin[\omega(t-r/c)]\right\}$$