Problem 4.13) The bound current-density of the magnetic point-dipole $m\delta(\boldsymbol{r})\hat{\boldsymbol{z}}$ is given by

$$\boldsymbol{J}_{\text{bound}}^{(e)} = \mu_0^{-1}\boldsymbol{\nabla}\times\boldsymbol{M}(\boldsymbol{r}) = \mu_0^{-1}\boldsymbol{\nabla}\times[m\delta(x)\delta(y)\delta(z)\hat{\boldsymbol{z}}]$$
$$= (m/\mu_0)[\delta(x)\delta'(y)\delta(z)\hat{\boldsymbol{x}} - \delta'(x)\delta(y)\delta(z)\hat{\boldsymbol{y}}].$$

One way to solve for the H-field is to write Maxwell's 2nd and 4th equations, the only ones that are relevant to magnetostatics, as follows:

Eq.(2): $\boldsymbol{\nabla}\times\boldsymbol{H}(\boldsymbol{r}) = \boldsymbol{J}_{\text{free}} + \partial\boldsymbol{D}/\partial t = 0 \;\rightarrow\; \mathrm{i}\boldsymbol{k}\times\boldsymbol{H}(\boldsymbol{k}) = 0 \;\rightarrow\;$ $\boldsymbol{H}(\boldsymbol{k})$ is a longitudinal field.

Eq.(4): $\boldsymbol{\nabla}\cdot\boldsymbol{B}(\boldsymbol{r}) = 0 \;\rightarrow\; \boldsymbol{\nabla}\cdot(\mu_0\boldsymbol{H}+\boldsymbol{M}) = 0 \;\rightarrow\; \mu_0\boldsymbol{\nabla}\cdot\boldsymbol{H}(\boldsymbol{r}) = -\boldsymbol{\nabla}\cdot\boldsymbol{M}(\boldsymbol{r})$

$\rightarrow\; \mathrm{i}\mu_0\boldsymbol{k}\cdot\boldsymbol{H}(\boldsymbol{k}) = -\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{M}(\boldsymbol{k}) \;\rightarrow\; \mu_0\boldsymbol{k}\cdot\boldsymbol{H}(\boldsymbol{k}) = -\boldsymbol{k}\cdot m\hat{\boldsymbol{z}} \;\rightarrow\; \boldsymbol{H}(\boldsymbol{k}) = -(m/\mu_0)k_z\hat{\boldsymbol{k}}/k.$

The function $\boldsymbol{H}(\boldsymbol{k})$ must now be inverse Fourier transformed to the space domain, as follows:

$$\boldsymbol{H}(\boldsymbol{r}) = \mathcal{F}^{-1}\{\boldsymbol{H}(\boldsymbol{k})\} = -(2\pi)^{-3}(m/\mu_0)\iiint_{-\infty}^{\infty}(k_z\hat{\boldsymbol{k}}/k)\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})\,d\boldsymbol{k}$$

$$= \mathrm{i}\left(\frac{m}{8\pi^3\mu_0}\right)\times\frac{\partial}{\partial z}\iiint_{-\infty}^{\infty}(\boldsymbol{k}/k^2)\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})\,d\boldsymbol{k} \quad \leftarrow \boxed{k_z\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}) = -\mathrm{i}(\partial/\partial z)\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})}$$

$$= \mathrm{i}\left(\frac{m}{8\pi^3\mu_0}\right)\times\frac{\partial}{\partial z}\int_{k=0}^{\infty}\int_{\theta=0}^{\pi}(k\cos\theta/k^2)\hat{\boldsymbol{r}}\exp(\mathrm{i}kr\cos\theta)\,2\pi k^2\sin\theta\,d\theta dk$$

$$= \mathrm{i}\left(\frac{m}{4\pi^2\mu_0}\right)\times\frac{\partial}{\partial z}\int_{k=0}^{\infty}k\hat{\boldsymbol{r}}\left[\int_{\theta=0}^{\pi}\sin\theta\cos\theta\exp(\mathrm{i}kr\cos\theta)\,d\theta\right]dk \quad \leftarrow \boxed{\text{G\&R } \mathbf{3.715}\text{-11}}$$

$$= \mathrm{i}^2\left(\frac{m}{2\pi^2\mu_0}\right)\times\frac{\partial}{\partial z}\int_{k=0}^{\infty}k\hat{\boldsymbol{r}}\{[\sin(kr) - kr\cos(kr)]/(kr)^2\}\,dk \leftarrow \boxed{\text{Change of variable: } x = kr}$$

$$= -\left(\frac{m}{2\pi^2\mu_0}\right)\times\frac{\partial}{\partial z}\left(\frac{\hat{\boldsymbol{r}}}{r^2}\right)\left[\underbrace{\int_0^{\infty}(\sin x/x)dx}_{\frac{1}{2}\pi} - \underbrace{\int_0^{\infty}\cos x\,dx}_{0}\right]$$

$$= -\frac{m}{4\pi\mu_0}\times\frac{\partial}{\partial z}\left(\frac{\hat{\boldsymbol{r}}}{r^2}\right) = -\frac{m}{4\pi\mu_0}\times\frac{\partial}{\partial z}\left[\frac{x\hat{\boldsymbol{x}}+y\hat{\boldsymbol{y}}+z\hat{\boldsymbol{z}}}{(x^2+y^2+z^2)^{3/2}}\right] = -\frac{m}{4\pi\mu_0}\times\frac{r^3\hat{\boldsymbol{z}} - 3zr^2\hat{\boldsymbol{r}}}{r^6}$$

$$= -\frac{m}{4\pi\mu_0}\times\frac{(\cos\theta\hat{\boldsymbol{r}} - \sin\theta\hat{\boldsymbol{\theta}}) - 3\cos\theta\hat{\boldsymbol{r}}}{r^3} = \frac{m(2\cos\theta\hat{\boldsymbol{r}} + \sin\theta\hat{\boldsymbol{\theta}})}{4\pi\mu_0 r^3}.$$

An alternative method of solving this problem involves writing Maxwell's 2nd and 4th equations as follows:

Eq.(4): $\boldsymbol{\nabla}\cdot\boldsymbol{B}(\boldsymbol{r}) = 0 \;\rightarrow\; \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{B}(\boldsymbol{k}) = 0 \;\rightarrow\;$ $\boldsymbol{B}(\boldsymbol{k})$ is a transverse field.

Eq.(2): $\boldsymbol{\nabla}\times\boldsymbol{H}(\boldsymbol{r}) = 0 \;\rightarrow\; \boldsymbol{\nabla}\times(\mu_0\boldsymbol{H}+\boldsymbol{M}) = \boldsymbol{\nabla}\times\boldsymbol{M}(\boldsymbol{r}) \;\rightarrow\; \mathrm{i}\boldsymbol{k}\times\boldsymbol{B}(\boldsymbol{k}) = \mathrm{i}\boldsymbol{k}\times\boldsymbol{M}(\boldsymbol{k})$

$\rightarrow\; \boldsymbol{k}\times\boldsymbol{B}(\boldsymbol{k}) = \boldsymbol{k}\times m\hat{\boldsymbol{z}} \;\rightarrow\; \boldsymbol{k}\times[\boldsymbol{k}\times\boldsymbol{B}(\boldsymbol{k})] = \boldsymbol{k}\times(\boldsymbol{k}\times m\hat{\boldsymbol{z}})$

$\rightarrow\; [\underbrace{\boldsymbol{k}\cdot\boldsymbol{B}(\boldsymbol{k})}_{0}]\boldsymbol{k} - (\boldsymbol{k}\cdot\boldsymbol{k})\boldsymbol{B}(\boldsymbol{k}) = m(\boldsymbol{k}\cdot\hat{\boldsymbol{z}})\boldsymbol{k} - m(\boldsymbol{k}\cdot\boldsymbol{k})\hat{\boldsymbol{z}} \;\rightarrow\; \boldsymbol{B}(\boldsymbol{k}) = -(mk_z/k^2)\boldsymbol{k} + m\hat{\boldsymbol{z}}.$

Given that $\boldsymbol{B}(\boldsymbol{k}) = \mu_0\boldsymbol{H}(\boldsymbol{k}) + \boldsymbol{M}(\boldsymbol{k}) = \mu_0\boldsymbol{H}(\boldsymbol{k}) + m\hat{\boldsymbol{z}}$, we will have $\boldsymbol{H}(\boldsymbol{k}) = -(m/\mu_0)k_z\hat{\boldsymbol{k}}/k$, which is the same result as obtained previously.