Problem 4.12) We begin by computing the Fourier transform of the uniformly-polarized spherical particle, as follows:

$$\boldsymbol{P}(\boldsymbol{k}) = \iiint_{-\infty}^{\infty} \boldsymbol{P}(\boldsymbol{r}) \exp(-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})\,\mathrm{d}\boldsymbol{r} = \int_{r=0}^{R}\int_{\theta=0}^{\pi} P_0\hat{\boldsymbol{z}} \exp(-\mathrm{i}kr\cos\theta)\, 2\pi r^2 \sin\theta\,\mathrm{d}r\mathrm{d}\theta$$

$$= 2\pi P_0\hat{\boldsymbol{z}} \int_{r=0}^{R} r^2\Big[\int_{\theta=0}^{\pi} \sin\theta \exp(-\mathrm{i}kr\cos\theta)\,\mathrm{d}\theta\Big]\mathrm{d}r$$

$$= 2\pi P_0\hat{\boldsymbol{z}} \int_0^R r^2 \left[\frac{\exp(-\mathrm{i}kr)-\exp(\mathrm{i}kr)}{-\mathrm{i}kr}\right]\mathrm{d}r = (4\pi P_0\hat{\boldsymbol{z}}/k)\int_0^R r\sin(kr)\,\mathrm{d}r \leftarrow \text{Change of variable: } x = kr$$

$$\text{Integration by parts} \rightarrow = (4\pi P_0\hat{\boldsymbol{z}}/k^3)\int_0^{kR} x\sin(x)\,\mathrm{d}x = (4\pi P_0\hat{\boldsymbol{z}}/k^3)\left[-x\cos x|_0^{kR} + \int_0^{kR}\cos x\,\mathrm{d}x\right]$$

$$= 4\pi P_0\hat{\boldsymbol{z}}\,[\sin(kR) - kR\cos(kR)]/k^3. \qquad (1)$$

Given that $\rho_{\text{bound}}^{(e)}(\boldsymbol{r},t) = -\boldsymbol{\nabla}\cdot\boldsymbol{P}(\boldsymbol{r},t)$, in the Fourier domain we will have

$$\rho_{\text{bound}}^{(e)}(\boldsymbol{k}) = -\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{P}(\boldsymbol{k}) = -\,\mathrm{i}4\pi P_0(\hat{\boldsymbol{z}}\cdot\hat{\boldsymbol{k}})[\sin(kR) - kR\cos(kR)]/k^2. \qquad (2)$$

Now, the scalar potential in the Fourier domain is known to be

$$\psi(\boldsymbol{k}) = \rho_{\text{total}}^{(e)}(\boldsymbol{k})/(\varepsilon_0 k^2). \qquad (3)$$

Therefore, inverse Fourier transformation of $\psi(\boldsymbol{k})$ yields

$$\psi(\boldsymbol{r}) = -\frac{\mathrm{i}4\pi P_0}{(2\pi)^3\varepsilon_0}\hat{\boldsymbol{z}}\cdot\int_{-\infty}^{\infty}\hat{\boldsymbol{k}}\left[\frac{\sin(kR)-kR\cos(kR)}{k^4}\right]\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})\,\mathrm{d}\boldsymbol{k}$$

$$= -\frac{\mathrm{i}P_0}{2\pi^2\varepsilon_0}(\hat{\boldsymbol{z}}\cdot\hat{\boldsymbol{r}})\int_{k=0}^{\infty}\int_{\theta=0}^{\pi}\cos\theta\left[\frac{\sin(kR)-kR\cos(kR)}{k^4}\right]\exp(\mathrm{i}kr\cos\theta)\,2\pi k^2\sin\theta\,\mathrm{d}\theta\mathrm{d}k$$

$$= -\frac{\mathrm{i}P_0}{\pi\varepsilon_0}(\hat{\boldsymbol{z}}\cdot\hat{\boldsymbol{r}})\int_{k=0}^{\infty}\left[\frac{\sin(kR)-kR\cos(kR)}{k^2}\right]\int_{\theta=0}^{\pi}\sin\theta\cos\theta\exp(\mathrm{i}kr\cos\theta)\,\mathrm{d}\theta\mathrm{d}k$$

$$= \frac{2P_0}{\pi\varepsilon_0 r^2}(\hat{\boldsymbol{z}}\cdot\hat{\boldsymbol{r}})\int_0^{\infty}\frac{\sin(kR)-kR\cos(kR)}{k^2}\times\frac{\sin(kr)-kr\cos(kr)}{k^2}\,\mathrm{d}k \leftarrow \text{G\&R } \mathbf{3.715}\text{-11}$$

$$= \frac{2P_0}{\pi\varepsilon_0 r^2}(\hat{\boldsymbol{z}}\cdot\hat{\boldsymbol{r}})\int_0^{\infty}\frac{\mathrm{d}}{\mathrm{d}k}[\sin(kR)/k]\times\frac{\mathrm{d}}{\mathrm{d}k}[\sin(kr)/k]\mathrm{d}k. \qquad (4)$$

To evaluate the integral appearing in the above equation, note that $\sin(\alpha k)/k$ is the Fourier transform of ½Rect$(x/2\alpha)$, which implies that $\mathrm{d}[\sin(\alpha k)/k]/\mathrm{d}k$ is the Fourier transform of $-(\mathrm{i}x/2)\mathrm{Rect}(x/2\alpha)$. It is not difficult to show that $\int_{-\infty}^{\infty}F(k)G(k)\mathrm{d}k = 2\pi\int_{-\infty}^{\infty}f(x)g(-x)\mathrm{d}x$ for any pair of functions $f(x)$ and $g(x)$ whose (one-dimensional) Fourier transforms are specified as $F(k)$ and $G(k)$. Finally, taking note of the fact that the integrand in Eq.(4) is an even function of k, we write

$$\int_0^{\infty}\frac{\mathrm{d}}{\mathrm{d}k}[\sin(kR)/k]\times\frac{\mathrm{d}}{\mathrm{d}k}[\sin(kr)/k]\mathrm{d}k = \pi\int_{-\infty}^{\infty}(-\mathrm{i}x/2)\mathrm{Rect}(x/2R)\times(\mathrm{i}x/2)\mathrm{Rect}(-x/2r)\mathrm{d}x$$

$$= (\pi/4)\int_{-\min(r,R)}^{\min(r,R)} x^2\mathrm{d}x = (\pi/6)\min(r^3,R^3). \qquad (5)$$

Upon substitution from Eq.(5) into Eq.(4), and also replacing $\hat{\boldsymbol{z}} \cdot \hat{\boldsymbol{r}}$ with $\cos\theta$, we find

$$\psi(\boldsymbol{r}) = \begin{cases} P_0 r\cos\theta/(3\varepsilon_0)\,; & r \le R, \\ P_0 R^3\cos\theta/(3\varepsilon_0 r^2)\,; & r \ge R. \end{cases} \tag{6}$$

Inside the spherical particle, $\psi(\boldsymbol{r}) = (P_0/3\varepsilon_0)z$. Therefore,

$$\boldsymbol{E}(\boldsymbol{r}) = -\boldsymbol{\nabla}\psi(\boldsymbol{r}) = -(\partial\psi/\partial z)\hat{\boldsymbol{z}} = -(P_0/3\varepsilon_0)\hat{\boldsymbol{z}}. \tag{7}$$

Outside the particle,

$$\boldsymbol{E}(\boldsymbol{r}) = -\boldsymbol{\nabla}\psi(\boldsymbol{r}) = -\frac{\partial\psi}{\partial r}\hat{\boldsymbol{r}} - \frac{\partial\psi}{r\partial\theta}\hat{\boldsymbol{\theta}} = \frac{(4\pi/3)R^3P_0}{4\pi\varepsilon_0 r^3}\left(2\cos\theta\,\hat{\boldsymbol{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}\right). \tag{8}$$

Thus, from outside the sphere, it appears as though a point-dipole $p_0\hat{\boldsymbol{z}} = (4\pi/3)R^3P_0\hat{\boldsymbol{z}}$ is residing at the origin of the coordinate system. Inside the particle, as can be seen from Eq.(7), the E-field is uniform and oriented opposite to the direction of polarization.

Digression: We prove below that $\int_{-\infty}^{\infty} F(k)G(k)\mathrm{d}k = 2\pi\int_{-\infty}^{\infty} f(x)g(-x)\mathrm{d}x$ for any pair of functions $f(x)$ and $g(x)$, whose (one-dimensional) Fourier transforms are specified as $F(k)$ and $G(k)$.

$$\int_{-\infty}^{\infty} F(k)G(k)\mathrm{d}k = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(x)\exp(-\mathrm{i}kx)\,\mathrm{d}x\right]G(k)\mathrm{d}k$$
$$= \int_{-\infty}^{\infty} f(x)\left[\int_{-\infty}^{\infty} G(k)\exp(-\mathrm{i}kx)\,\mathrm{d}k\right]\mathrm{d}x = 2\pi\int_{-\infty}^{\infty} f(x)g(-x)\mathrm{d}x. \tag{9}$$

Also, it is easy to show that, if $F(k)$ is the Fourier transform of $f(x)$, then $\mathrm{d}F(k)/\mathrm{d}k$ is the Fourier transform of $-\mathrm{i}xf(x)$. This is because differentiating $F(k) = \int_{-\infty}^{\infty} f(x)\exp(-\mathrm{i}kx)\,\mathrm{d}x$ with respect to k yields

$$\mathrm{d}F(k)/\mathrm{d}k = \int_{-\infty}^{\infty} f(x)(-\mathrm{i}x)\exp(-\mathrm{i}kx)\,\mathrm{d}x = \mathcal{F}\{-\mathrm{i}xf(x)\}. \tag{10}$$

We have used both Eq.(9) and Eq.(10) in evaluating the definite integral given by Eq.(5).