Problem 11)

a) $\rho(\vec{r}) = \rho_0 z\,\delta(r-R) \Rightarrow$

$$\rho(\vec{k}) = \rho_0 \int_{-\infty}^{\infty} z\,\delta(r-R)\, e^{-i\vec{k}\cdot\vec{r}} d\vec{r} = \rho_0 \int_{-\infty}^{\infty} z e^{-ik_z z} dz \int_{r=0}^{\infty}\int_{\theta=0}^{2\pi} \delta(r-R)\, e^{-ik_\parallel r\cos\theta}\, r\,d\theta\,dr$$

$$= \rho_0 \frac{\partial}{\partial k_z}\int_{-\infty}^{\infty} i e^{-ik_z z} dz \underbrace{\int_{r=0}^{\infty} 2\pi r\,\delta(r-R)\, J_0(k_\parallel r)\,dr} = i(2\pi)^2 \rho_0 R\, J_0(k_\parallel R)\,\delta'(k_z)$$

$$\psi(\vec{k}) = \frac{\rho(\vec{k})}{\epsilon_0 k^2} \Rightarrow \psi(\vec{r}) = \frac{i(2\pi)^2 \rho_0 R}{\epsilon_0 (2\pi)^3} \int_{-\infty}^{\infty} \frac{J_0(k_\parallel R)\,\delta'(k_z)}{k^2} e^{i\vec{k}\cdot\vec{r}} d\vec{k} \Rightarrow$$

$$\psi(\vec{r}) = \frac{i\rho_0 R}{2\pi\epsilon_0} \int_{k_\parallel=0}^{\infty}\int_{\theta=0}^{2\pi} J_0(k_\parallel R)\, e^{ik_\parallel r_\parallel \cos\theta} \left\{ \int_{k_z=-\infty}^{\infty} \delta'(k_z) \frac{e^{ik_z z}}{k_\parallel^2 + k_z^2} dk_z \right\} k_\parallel\, d\theta\, dk_\parallel$$

We now use the sifting property of $\delta'(\cdot)$ to evaluate the integral over k_z:

$$\int_{-\infty}^{\infty} \delta'(k_z) \frac{e^{ik_z z}}{k_\parallel^2 + k_z^2} dk_z = -\frac{\partial}{\partial k_z}\left(\frac{e^{ik_z z}}{k_\parallel^2 + k_z^2}\right)\Bigg|_{k_z=0} = -\frac{iz e^{ik_z z}(k_\parallel^2 + k_z^2) - 2k_z e^{ik_z z}}{(k_\parallel^2 + k_z^2)^2}\Bigg|_{k_z=0}$$

$$= -\frac{iz}{k_\parallel^2}$$

Therefore: $$\psi(\vec{r}) = \frac{\rho_0 R z}{\epsilon_0} \int_{k_\parallel=0}^{\infty} \frac{J_0(k_\parallel R) J_0(k_\parallel r_\parallel)}{k_\parallel} dk_\parallel \underset{\text{G.R. 6.533-4}}{=} \frac{\rho_0 R z}{\epsilon_0} \begin{cases} \int_0^{\infty} \frac{J_0(k_\parallel R)}{k_\parallel} dk_\parallel\,; & r_\parallel < R \\ -\ln\left(\frac{r_\parallel}{R}\right) + \int_0^{\infty} \frac{J_0(k_\parallel R)}{k_\parallel} dk_\parallel\,; & r_\parallel > R \end{cases}$$

Now, let us define a constant $c = \int_0^{\infty} \frac{J_0(k_\parallel R)}{k_\parallel} dk_\parallel = \int_0^{\infty} \frac{J_0(x)}{x} dx$. The integral is divergent, that is, $c = \infty$. However, this is an artifact of having an infinitely-long wire. The important point is that c, large as it may be, is not a function of $r_\parallel$ or z. We thus write:

$$\psi(\vec{r}) = \frac{\rho_0 R z}{\epsilon_0} \begin{cases} c\,; & r_\parallel < R \quad \leftarrow \text{inside the wire} \\ c - \ln(r_\parallel/R)\,; & r_\parallel > R \quad \leftarrow \text{outside} \end{cases}$$

$$\vec{E}(\vec{r}) = -\vec{\nabla}\psi(\vec{r}) = -\frac{\partial\psi}{\partial r_\parallel}\hat{r}_\parallel - \frac{\partial\psi}{\partial z}\hat{z} = -(\rho_0 R/\epsilon_0)\begin{cases} C\hat{z}; & r_\parallel < R \\ -(z/r_\parallel)\hat{r}_\parallel + [C - \ln(r_\parallel/R)]\hat{z}; & r_\parallel > R \end{cases}$$

Once again, note that in the above expression $C = \int_0^\infty x^{-1} J_0(x)\,dx$ is an infinitely large constant.

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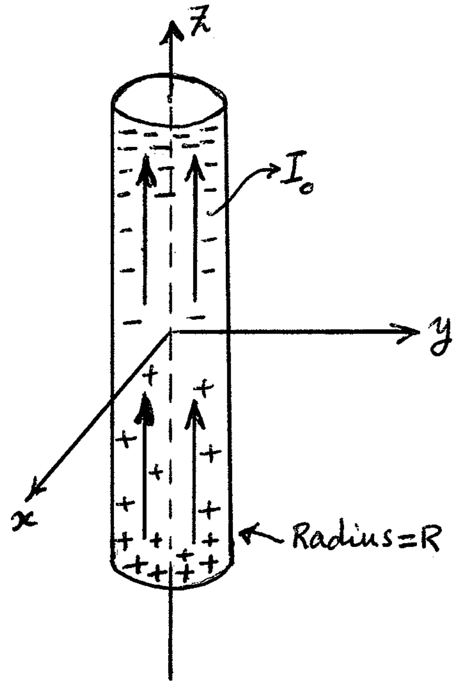
b) $\vec{H}(\vec{r}) = \frac{I_0}{2\pi r_\parallel}\hat{\phi};$ $\quad r_\parallel > R \leftarrow$ Outside the wire

$$\vec{S}(\vec{r}) = \vec{E}(\vec{r}) \times \vec{H}(\vec{r}) = -\left(\frac{\rho_0 R I_0}{2\pi\epsilon_0}\right)\left[-\left(\frac{z}{r_\parallel^2}\right)\hat{z} + \frac{\ln(r_\parallel/R) - C}{r_\parallel}\hat{r}_\parallel\right]; \quad r_\parallel > R$$

Now, suppose $I_0 > 0$, that is, the current flows in the $+z$ direction. Within the wire we must have $E_z > 0$; therefore, $\rho_0 < 0$, which means that the surface charge distribution is as shown in the figure.



The Poynting vector $\vec{S}(\vec{r})$ has a radial component that is directed toward the wire, namely, $\vec{S}_r = \left(\frac{-\rho_0 R I_0}{2\pi\epsilon_0}\right)\left(\frac{-C}{r_\parallel}\right)\hat{r}_\parallel$. This energy flows from $r_\parallel = \infty$ into the wire. Integrated over the circumference, its value is $(-\rho_0 R I_0/\epsilon_0)C$, independent of $r_\parallel$ and z coordinates. This energy gets converted to heat within the wire (ohmic loss). The larger the resistivity of the wire, the larger will be the magnitude of the induced-charge, ρ_0.

The radial component of $\vec{S}(\vec{r})$ also has a term that is directed away from the wire, namely, $\vec{S}_r = \left(\frac{-\rho_0 R I_0}{2\pi\epsilon_0}\right)\left[\frac{\ln(r_\parallel/R)}{r_\parallel}\right]\hat{r}_\parallel$. This represents the electromagnetic energy delivered to the "load" via the wire. Consider a hypothetical cylindrical surface of radius R' and height $\Delta z = 1^m$. Integrating the outward-directed component of $\vec{S}_r$ over this cylindrical surface yields $(-\rho_0 R I_0/\epsilon_0)\ln(R'/R)$ for the rate of flow of energy out of the system. The larger the chosen value of R', the greater will be the amount of energy that leaves the system. (This is yet another artifact of the infinite length of the wire.)

Next, consider the component of the Poynting vector along the z-axis, $\vec{S}_z(\vec{r}) = \left(\frac{-\rho_0 R I_0}{2\pi\epsilon_0}\right)\left(\frac{-z}{r_{\parallel}^2}\right)\hat{z}$. When $z<0$, the energy flows upward, whereas for $z>0$ the energy flow is downward. In either case, as $z\to 0$ the energy flux in the z-direction diminishes. This is because the z-directed energy gradually turns around and leaves the system in the radial direction. The integral of $S_z(\vec{r})$ over a cross-section of the system parallel to the xy-plane is given by:

$$\int_{r_{\parallel}=R}^{R'} S_z(\vec{r})\, 2\pi r_{\parallel}\, dr_{\parallel} = \left(\frac{-\rho_0 R I_0}{\epsilon_0}\right)\int_R^{R'}\left(\frac{-z}{r_{\parallel}^2}\right) r_{\parallel}\, dr_{\parallel} = (-\rho_0 R I_0/\epsilon_0)(-z)\ln(R'/R).$$

The change in the above value of z-directed energy flux over a distance $\Delta z = 1^m$ is precisely the energy flux out of the hypothetical cylindrical surface of radius R' and height $\Delta z = 1^m$ discussed earlier. The energy thus flows into the system at $z=\pm\infty$. As it moves vertically toward the xy-plane at $z=0$, it gradually turns around and leaves the system in the radially outward direction.

Digression: In practice, of course, the wire has a finite length along the z-axis, in which case the sign of ρ_0 does not change between the point at which the current enters the wire, and the point where it leaves the wire. Adding a uniform charge density to the surface of the wire also helps to control the magnitude of the z-directed energy flux, without changing the E-field inside the wire. The radial energy flux then serves mainly to feed the Ohmic losses of the wire itself.