Problem 10)

$$\vec{J}(\vec{r},t) = I_0\hat{z}\,\delta(x)\delta(y)\,\text{Rect}(z/L)$$

$$\vec{J}(\vec{k},\omega) = \int_{-\infty}^{\infty}\vec{J}(\vec{r},t)e^{-i(\vec{k}\cdot\vec{r}-\omega t)}d\vec{r}dt = I_0\hat{z}\int_{-\infty}^{\infty}\delta(x)e^{-ik_x x}dx\int_{-\infty}^{\infty}\delta(y)e^{-ik_y y}dy$$

$$\times\int_{-\infty}^{\infty}\text{Rect}(z/L)e^{-ik_z z}dz\int_{-\infty}^{\infty}e^{i\omega t}dt = I_0\hat{z}\,\frac{e^{-ik_z L/2}-e^{+ik_z L/2}}{-ik_z}\,2\pi\delta(\omega) \Rightarrow$$

$$\vec{J}(\vec{k},\omega) = 4\pi I_0\hat{z}\,\frac{\sin(Lk_z/2)}{k_z}\,\delta(\omega).$$

$$\vec{A}(\vec{r},t) = \frac{1}{(2\pi)^4}\int_{-\infty}^{\infty}\frac{\mu_0\vec{J}(\vec{k},\omega)}{k^2-(\omega/c)^2}e^{+i(\vec{k}\cdot\vec{r}-\omega t)}d\vec{k}d\omega$$

$$= \frac{4\pi\mu_0 I_0\hat{z}}{(2\pi)^4}\int_{-\infty}^{\infty}\frac{\sin(Lk_z/2)}{k_z(k^2-\frac{\omega^2}{c^2})}\delta(\omega)e^{+i(\vec{k}\cdot\vec{r}-\omega t)}d\vec{k}d\omega$$

$$= \frac{\mu_0 I_0\hat{z}}{4\pi^3}\int_{-\infty}^{\infty}\frac{\sin(Lk_z/2)}{k_z(k_x^2+k_y^2+k_z^2)}e^{i(k_x x+k_y y+k_z z)}dk_x dk_y dk_z$$

Let $\vec{k}_\parallel = k_x\hat{x}+k_y\hat{y}$ and $\vec{r}_\parallel = x\hat{x}+y\hat{y}$. We'll have:

$$\vec{A}(\vec{r},t) = \frac{\mu_0 I_0\hat{z}}{4\pi^3}\int_{-\infty}^{\infty}\frac{\sin(Lk_z/2)}{k_z}e^{ik_z z}\left\{\int_{k_\parallel=0}^{\infty}\int_{\theta=0}^{2\pi}\frac{e^{ik_\parallel r_\parallel\cos\theta}}{k_\parallel^2+k_z^2}k_\parallel dk_\parallel d\theta\right\}dk_z$$

G.R. 3.915-2

$$= \frac{\mu_0 I_0\hat{z}}{4\pi^3}\int_{-\infty}^{\infty}\frac{\sin(Lk_z/2)}{k_z}e^{ik_z z}\left\{\int_{k_\parallel=0}^{\infty}\frac{2\pi k_\parallel J_0(k_\parallel r_\parallel)}{k_\parallel^2+k_z^2}dk_\parallel\right\}dk_z$$

G.R. 6.532-4

$$= \frac{\mu_0 I_0\hat{z}}{2\pi^2}\int_{-\infty}^{\infty}\frac{\sin(Lk_z/2)}{k_z}e^{ik_z z}K_0(|k_z|r_\parallel)dk_z$$

$$= \frac{\mu_0 I_0\hat{z}}{2\pi^2}\left\{\int_{-\infty}^{0}\frac{\sin(Lk_z/2)}{k_z}e^{ik_z z}K_0(|k_z|r_\parallel)dk_z + \int_0^{\infty}\frac{\sin(Lk_z/2)}{k_z}e^{ik_z z}K_0(|k_z|r_\parallel)dk_z\right\}$$

$$= \frac{\mu_0 I_0\hat{z}}{\pi^2}\int_0^{\infty}\frac{\sin(Lk_z/2)\cos(k_z z)}{k_z}K_0(k_z r_\parallel)dk_z$$

$$= \frac{\mu_0 I_0\hat{z}}{2\pi^2}\int_0^{\infty}\frac{\sin[(\frac{L}{2}+z)k_z]+\sin[(\frac{L}{2}-z)k_z]}{k_z}K_0(k_z r_\parallel)dk_z$$

Using G.R. 6.699-3, 9.121-28, and 8.338-2, we'll have

$$\vec{A}(\vec{r},t) = \frac{\mu_0 I_0 \hat{z}}{4\pi}\left\{ \ln \frac{|\frac{L}{2}+z| + \sqrt{(\frac{L}{2}+z)^2 + r_{\parallel}^2}}{r_{\parallel}} + \ln \frac{|\frac{L}{2}-z| + \sqrt{(\frac{L}{2}-z)^2 + r_{\parallel}^2}}{r_{\parallel}} \right\}$$

$$= \frac{\mu_0 I_0 \hat{z}}{4\pi}\left\{ \ln\left[\left(|\tfrac{L}{2}+z| + \sqrt{(L/2+z)^2 + r_{\parallel}^2}\right)\left(|\tfrac{L}{2}-z| + \sqrt{(L/2-z)^2 + r_{\parallel}^2}\right)\right] - 2\ln r_{\parallel} \right\}$$

In the limit $L \to \infty$, the logarithmic term becomes infinitely large. However, its dependence on $r_{\parallel}$ and z diminishes to the point that any <u>variations</u> of this term with $r_{\parallel}$ and z may be ignored. In other words, when computing $\vec{\nabla}\times\vec{A}$, the derivatives with respect to $r_{\parallel}$ and z of the logarithmic term approach zero when $L \to \infty$. We thus ignore this logarithmic term; even though its magnitude is very large, it effectively behaves like a <u>constant</u> term. We thus have

$$\vec{A}(\vec{r},t) = -\frac{\mu_0 I_0 \hat{z}}{2\pi} \ln r_{\parallel} + \underset{(\infty)}{\text{Constant}} \leftarrow (\text{when } L \to \infty)$$

$$\vec{B}(\vec{r},t) = \vec{\nabla}\times\vec{A}(\vec{r},t) \Rightarrow \mu_0 \vec{H}(\vec{r},t) = -\frac{\partial A_z}{\partial r_{\parallel}}\hat{\phi} = \frac{\mu_0 I_0}{2\pi r_{\parallel}}\hat{\phi} \Rightarrow \vec{H}(\vec{r},t) = \frac{I_0 \hat{\phi}}{2\pi r_{\parallel}}.$$