Problem 8) The circular loop's current density $\boldsymbol{J}(\boldsymbol{r},t)=I_o\delta(r_{||}-R)\,\delta(z)\hat{\boldsymbol{\phi}}$ may be Fourier transformed as follows:

$$\boldsymbol{J}(\boldsymbol{k},\omega)=\iiiint_{-\infty}^{\infty}\boldsymbol{J}(\boldsymbol{r},t)\exp[-\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\,\mathrm{d}\boldsymbol{r}\,\mathrm{d}t=2\pi I_o\delta(\omega)\hat{\boldsymbol{z}}\times\iint_{-\infty}^{\infty}\hat{\boldsymbol{r}}_{||}\,\delta(r_{||}-R)\exp(-\mathrm{i}\boldsymbol{k}_{||}\cdot\boldsymbol{r}_{||})\,\mathrm{d}\boldsymbol{r}_{||}$$

$$=2\pi I_o\delta(\omega)(\hat{\boldsymbol{z}}\times\hat{\boldsymbol{k}}_{||})\iint_{-\infty}^{\infty}\cos\phi\,\delta(r_{||}-R)\exp(-\mathrm{i}k_{||}r_{||}\cos\phi)\,r_{||}\,\mathrm{d}\phi\,\mathrm{d}r_{||}$$

$$=2\pi I_o\delta(\omega)(\hat{\boldsymbol{z}}\times\hat{\boldsymbol{k}}_{||})\int_0^{\infty}r_{||}\,\delta(r_{||}-R)\,\mathrm{d}r_{||}\int_0^{2\pi}\cos\phi\exp(-\mathrm{i}k_{||}r_{||}\cos\phi)\,\mathrm{d}\phi$$

$$=-\mathrm{i}(2\pi)^2RI_oJ_1(Rk_{||})\,\delta(\omega)(\hat{\boldsymbol{z}}\times\hat{\boldsymbol{k}}_{||})$$

← $J_1(\cdot)$ is Bessel function of first kind, 1st order. (G&R 3.915-2)

Inverse Fourier transforming the vector potential $\boldsymbol{A}(\boldsymbol{k},\omega)=\mu_o\boldsymbol{J}(\boldsymbol{k},\omega)/(k^2-\omega^2/c^2)$ then yields,

$$\boldsymbol{A}(\boldsymbol{r},t)=(2\pi)^{-4}\iiiint_{-\infty}^{\infty}\boldsymbol{A}(\boldsymbol{k},\omega)\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\,\mathrm{d}\boldsymbol{k}\,\mathrm{d}\omega$$

$$=-\mathrm{i}(2\pi)^{-2}\mu_oRI_o\hat{\boldsymbol{z}}\times\iiint_{-\infty}^{\infty}\hat{\boldsymbol{k}}_{||}[J_1(Rk_{||})\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})/(k_{||}^2+k_z^2)]\,\mathrm{d}\boldsymbol{k}$$

$$=-\mathrm{i}(\mu_oRI_o/4\pi)\hat{\boldsymbol{z}}\times\iint_{-\infty}^{\infty}(\hat{\boldsymbol{k}}_{||}/k_{||})\exp(-|z|k_{||})\,J_1(Rk_{||})\exp(\mathrm{i}\boldsymbol{k}_{||}\cdot\boldsymbol{r}_{||})\,\mathrm{d}\boldsymbol{k}_{||}$$ (G&R 3.389-5)

$$=-\mathrm{i}(\mu_oRI_o/4\pi)(\hat{\boldsymbol{z}}\times\hat{\boldsymbol{r}}_{||})\int_0^{\infty}\exp(-|z|k_{||})\,J_1(Rk_{||})\,\mathrm{d}k_{||}\int_0^{2\pi}\cos\phi\exp(\mathrm{i}k_{||}r_{||}\cos\phi)\,\mathrm{d}\phi$$

$$=\tfrac{1}{2}\mu_oRI_o\hat{\boldsymbol{\phi}}\int_0^{\infty}\exp(-|z|k_{||})\,J_1(Rk_{||})\,J_1(r_{||}k_{||})\,\mathrm{d}k_{||}$$ (G&R 3.915-2)

$$=(\mu_oI_o/2\pi)\sqrt{R/r_{||}}\,Q_{1/2}[(R^2+r_{||}^2+z^2)/(2Rr_{||})]\hat{\boldsymbol{\phi}}$$

← $Q_{1/2}(\cdot)$ is Legendre function of 2nd kind, order ½. (G&R 6.612-3)

Alternatively, one may calculate the vector potential $\boldsymbol{A}(\boldsymbol{r},t)$ by direct integration over the current density of the loop, namely,

$$\boldsymbol{A}(\boldsymbol{r},t)=(\mu_o/4\pi)\iiint_{-\infty}^{\infty}[\boldsymbol{J}(\boldsymbol{r}',t-|\boldsymbol{r}-\boldsymbol{r}'|/c)/|\boldsymbol{r}-\boldsymbol{r}'|]\,\mathrm{d}\boldsymbol{r}'$$

$$=(\mu_oRI_o/2\pi)\hat{\boldsymbol{\phi}}\int_0^{\pi}[\cos\phi/\sqrt{R^2+r_{||}^2+z^2-2Rr_{||}\cos\phi}]\,\mathrm{d}\phi.$$

The two methods must yield identical results; therefore, defining $x=(R^2+r_{||}^2+z^2)/(2Rr_{||})$, where $x>1$, we conclude that the following identity must hold:

$$\sqrt{2}\,Q_{1/2}(x)=\int_0^{\pi}(\cos\phi/\sqrt{x-\cos\phi})\,\mathrm{d}\phi$$

$$=x\int_0^{\pi}\mathrm{d}\phi/\sqrt{x-\cos\phi}-\int_0^{\pi}\sqrt{x-\cos\phi}\,\mathrm{d}\phi$$

$$=2(x/\sqrt{x+1})K[\sqrt{2/(x+1)}]-2\sqrt{x+1}\,E[\sqrt{2/(x+1)}].$$ (G&R 3.671-4, 5)

In the last equation, $K(\cdot)$ and $E(\cdot)$ are complete elliptic integrals of the first and second kind, respectively. For a description of these and other elliptic integrals see G&R 8.1.

In the limit when $R\to 0$, the current loop must approach a magnetic point-dipole. In this limit $x\to\infty$, and the arguments of $K(\cdot)$ and $E(\cdot)$ approach zero. We thus use the first three terms in the Taylor series expansions of $K(\cdot)$ and $E(\cdot)$, namely, $K(k)=\tfrac{1}{2}\pi[1+(k^2/4)+(9k^4/64)+\cdots]$ and $E(k)=\tfrac{1}{2}\pi[1-(k^2/4)-(3k^4/64)-\cdots]$ to write ← G&R 8.113-1 and 8.114-1

$$\begin{aligned}\boldsymbol{A}(\boldsymbol{r},t)&=(\mu_o I_o/2\pi)\sqrt{2R/r_{||}}\,\{(x/\sqrt{x+1})K[\sqrt{2/(x+1)}]-\sqrt{x+1}\,E[\sqrt{2/(x+1)}]\}\hat{\boldsymbol{\phi}}\\ &\approx(\mu_o I_o/16)\sqrt{2R/r_{||}}\,[(x+1)^{-3/2}-(9/4)(x+1)^{-5/2}]\hat{\boldsymbol{\phi}}\\ &\approx(\mu_o I_o/16)\sqrt{2R/r_{||}}\,\{2Rr_{||}/[(R+r_{||})^2+z^2]\}^{3/2}\hat{\boldsymbol{\phi}}\\ &\approx(\mu_o I_o R^2/4)r_{||}\hat{\boldsymbol{\phi}}/(r_{||}{}^2+z^2)^{3/2}\\ &\approx(\mu_o/4\pi)\pi R^2 I_o\hat{\boldsymbol{z}}\times\hat{\boldsymbol{r}}/r^2.\end{aligned}$$

The last expression is the vector potential of the magnetic point-dipole $\boldsymbol{m}=\mu_o(\pi R^2)I_o\hat{\boldsymbol{z}}$.