Problem 7)

a) Charge density $\rho(\boldsymbol{r},t)=\sigma_{so}\mathrm{Circ}(r_{||}/R)\delta(z)$; current density $\boldsymbol{J}(\boldsymbol{r},t)=\sigma_{so}r_{||}\Omega\,\mathrm{Circ}(r_{||}/R)\delta(z)\hat{\boldsymbol{\phi}}$. Here we have taken into account the velocity of the charges being $r_{||}\Omega$ at a location on the disk where the radius is $r_{||}$.

b) $$\rho(\boldsymbol{k},\omega)=\iiiint_{-\infty}^{\infty}\rho(\boldsymbol{r},t)\exp[-\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\,\mathrm{d}\boldsymbol{r}\,\mathrm{d}t$$

$$=2\pi\sigma_{so}\delta(\omega)\iint_{-\infty}^{\infty}\mathrm{Circ}(r_{||}/R)\exp(-\mathrm{i}\boldsymbol{k}_{||}\cdot\boldsymbol{r}_{||})\,\mathrm{d}\boldsymbol{r}_{||}$$

$$=2\pi\sigma_{so}\delta(\omega)\iint_{-\infty}^{\infty}\mathrm{Circ}(r_{||}/R)\exp(-\mathrm{i}k_{||}r_{||}\cos\phi)\,r_{||}\,\mathrm{d}\phi\,\mathrm{d}r_{||}$$

$$=2\pi\sigma_{so}\delta(\omega)\int_0^R r_{||}\,\mathrm{d}r_{||}\int_0^{2\pi}\exp(-\mathrm{i}k_{||}r_{||}\cos\phi)\,\mathrm{d}\phi$$

$$=(2\pi)^2\sigma_{so}\delta(\omega)\int_0^R r_{||}J_0(k_{||}r_{||})\,\mathrm{d}r_{||}$$ ← $J_0(\cdot)$ is Bessel function of first kind, 0th order

$$=(2\pi)^2\sigma_{so}Rk_{||}^{-1}J_1(Rk_{||})\delta(\omega).$$ ← $J_1(\cdot)$ is Bessel function of first kind, 1st order (G&R 3.915-2 and **5.52**−1)

$$\boldsymbol{J}(\boldsymbol{k},\omega)=\iiiint_{-\infty}^{\infty}\boldsymbol{J}(\boldsymbol{r},t)\exp[-\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\,\mathrm{d}\boldsymbol{r}\,\mathrm{d}t$$

$$=2\pi\sigma_{so}\Omega\delta(\omega)\hat{\boldsymbol{z}}\times\iint_{-\infty}^{\infty}\hat{\boldsymbol{r}}_{||}\,r_{||}\mathrm{Circ}(r_{||}/R)\exp(-\mathrm{i}\boldsymbol{k}_{||}\cdot\boldsymbol{r}_{||})\,\mathrm{d}\boldsymbol{r}_{||}$$

$$=2\pi\sigma_{so}\Omega\delta(\omega)\hat{\boldsymbol{z}}\times\iint_{-\infty}^{\infty}\hat{\boldsymbol{k}}_{||}\cos\phi\,\mathrm{Circ}(r_{||}/R)\exp(-\mathrm{i}k_{||}r_{||}\cos\phi)\,r_{||}^2\,\mathrm{d}\phi\,\mathrm{d}r_{||}$$

$$=2\pi\sigma_{so}\Omega\delta(\omega)(\hat{\boldsymbol{z}}\times\hat{\boldsymbol{k}}_{||})\int_0^R r_{||}^2\,\mathrm{d}r_{||}\int_0^{2\pi}\cos\phi\exp(-\mathrm{i}k_{||}r_{||}\cos\phi)\,\mathrm{d}\phi$$

$$=-\mathrm{i}(2\pi)^2\sigma_{so}\Omega\delta(\omega)(\hat{\boldsymbol{z}}\times\hat{\boldsymbol{k}}_{||})\int_0^R r_{||}^2J_1(k_{||}r_{||})\,\mathrm{d}r_{||}$$ ← $J_1(\cdot)$ is Bessel function of first kind, 1st order

$$=-\mathrm{i}(2\pi)^2\sigma_{so}\Omega R^2k_{||}^{-1}J_2(Rk_{||})\delta(\omega)(\hat{\boldsymbol{z}}\times\hat{\boldsymbol{k}}_{||}).$$ ← $J_2(\cdot)$ is Bessel function of first kind, 2nd order (G&R 3.915-2 and **5.52**−1)

c) Scalar potential $\psi(\boldsymbol{k},\omega)=\varepsilon_o^{-1}\rho(\boldsymbol{k},\omega)/(k^2-\omega^2/c^2)$;

Vector potential $\boldsymbol{A}(\boldsymbol{k},\omega)=\mu_o\boldsymbol{J}(\boldsymbol{k},\omega)/(k^2-\omega^2/c^2)$.

Digression: Inverse Fourier transforming the above potentials yields expressions for $\psi(\boldsymbol{r},t)$ and $\boldsymbol{A}(\boldsymbol{r},t)$ in the form of one-dimensional integrals, although no simple closed forms seem to exist.

$$\psi(\boldsymbol{r},t)=(2\pi)^{-4}\iiiint_{-\infty}^{\infty}\psi(\boldsymbol{k},\omega)\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\,\mathrm{d}\boldsymbol{k}\,\mathrm{d}\omega$$

$$=(2\pi)^{-2}\varepsilon_o^{-1}\sigma_{so}R\iiint_{-\infty}^{\infty}k_{||}^{-1}J_1(Rk_{||})\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})/(k_{||}^2+k_z^2)\,\mathrm{d}\boldsymbol{k}$$

$$=(\sigma_{so}R/4\pi\varepsilon_o)\iint_{-\infty}^{\infty}k_{||}^{-2}\exp(-|z|k_{||})J_1(Rk_{||})\exp(\mathrm{i}\boldsymbol{k}_{||}\cdot\boldsymbol{r}_{||})\,\mathrm{d}\boldsymbol{k}_{||}$$ (G&R 3.389-5)

$$=(\sigma_{so}R/4\pi\varepsilon_o)\int_0^{\infty}k_{||}^{-1}\exp(-|z|k_{||})J_1(Rk_{||})\,\mathrm{d}k_{||}\int_0^{2\pi}\exp(\mathrm{i}k_{||}r_{||}\cos\phi)\,\mathrm{d}\phi$$

$$= \tfrac{1}{2}(\sigma_{so} R/\varepsilon_o)\int_0^{\infty} k_{||}^{-1} \exp(-|z|k_{||})\, J_1(Rk_{||})\, J_0(r_{||}k_{||})\,\mathrm{d}k_{||}. \qquad \text{(G\&R 3.915-2)}$$

$$\boldsymbol{A}(\boldsymbol{r},t) = (2\pi)^{-4}\iiiint_{-\infty}^{\infty} \boldsymbol{A}(\boldsymbol{k},\omega)\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\,\mathrm{d}\boldsymbol{k}\,\mathrm{d}\omega$$

$$= -\mathrm{i}(2\pi)^{-2}\mu_o\sigma_{so}\Omega R^2\,\hat{\boldsymbol{z}}\times\iiint_{-\infty}^{\infty}(\hat{\boldsymbol{k}}_{||}/k_{||})[J_2(Rk_{||})\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})/(k_{||}^{\,2}+k_z^{\,2})]\,\mathrm{d}\boldsymbol{k}$$

$$= -\mathrm{i}(\mu_o\sigma_{so}\Omega R^2/4\pi)\hat{\boldsymbol{z}}\times\iint_{-\infty}^{\infty}(\hat{\boldsymbol{k}}_{||}/k_{||}^{\,2})\exp(-|z|k_{||})\, J_2(Rk_{||})\exp(\mathrm{i}\boldsymbol{k}_{||}\cdot\boldsymbol{r}_{||})\,\mathrm{d}\boldsymbol{k}_{||} \qquad \text{(G\&R 3.389-5)}$$

$$= -\mathrm{i}(\mu_o\sigma_{so}\Omega R^2/4\pi)(\hat{\boldsymbol{z}}\times\hat{\boldsymbol{r}}_{||})\int_0^{\infty} k_{||}^{-1}\exp(-|z|k_{||})\, J_2(Rk_{||})\,\mathrm{d}k_{||}\int_0^{2\pi}\cos\phi\,\exp(\mathrm{i}k_{||}r_{||}\cos\phi)\,\mathrm{d}\phi$$

$$= \tfrac{1}{2}\mu_o\sigma_{so}\Omega R^2\hat{\boldsymbol{\phi}}\int_0^{\infty} k_{||}^{-1}\exp(-|z|k_{||})\, J_2(Rk_{||})\, J_1(r_{||}k_{||})\,\mathrm{d}k_{||}. \qquad \text{(G\&R 3.915-2)}$$