Problem 6)

First we find the scalar potential function for the point-charge q_o:

$$\rho(\vec{r}) = q_o\,\delta(x)\delta(y)\delta(z-z_o) \Rightarrow \rho(\vec{k}) = \int_{-\infty}^{\infty} q_o\,\delta(x)\delta(y)\delta(z-z_o)\, e^{-i\vec{k}\cdot\vec{r}}\, d\vec{r}$$

$$\Rightarrow \rho(\vec{k}) = q_o e^{-ik_z z_o}$$

$$\psi(\vec{r}) = \frac{q_o}{\epsilon_o(2\pi)^3}\int_{-\infty}^{\infty} \frac{e^{-ik_z z_o}}{k_x^2+k_y^2+k_z^2}\, e^{+i\vec{k}\cdot\vec{r}}\, d\vec{k} = \frac{q_o}{\epsilon_o(2\pi)^3}\int_{k_z=-\infty}^{\infty}\int_{k_\parallel=0}^{\infty}\int_{\theta=0}^{2\pi} \frac{e^{ik_z(z-z_o)}}{k_\parallel^2+k_z^2}\times$$

$$e^{ik_\parallel r_\parallel \cos\theta}\, k_\parallel\, d\theta\, dk_\parallel = \frac{q_o}{\epsilon_o(2\pi)^2}\int_{k_z=-\infty}^{\infty} e^{ik_z(z-z_o)}\left\{\int_{k_\parallel=0}^{\infty} \frac{k_\parallel J_o(k_\parallel r_\parallel)}{k_\parallel^2+k_z^2}\, dk_\parallel\right\} dk_z$$

$$\overset{\text{G.R. 6.532-4}}{=} \frac{q_o}{\epsilon_o(2\pi)^2}\int_{k_z=-\infty}^{\infty} e^{ik_z(z-z_o)}\, K_o(r_\parallel |k_z|)\, dk_z = \frac{q_o}{2\pi^2\epsilon_o}\int_0^{\infty}\cos[k_z(z-z_o)]\,K_o(r_\parallel k_z)\, dk_z$$

$$\overset{\text{G.R. 6.671-14}}{=} \frac{q_o}{4\pi\epsilon_o}\,\frac{1}{\sqrt{r_\parallel^2+(z-z_o)^2}} \Rightarrow \psi(\vec{r}) = \frac{q_o}{4\pi\epsilon_o\sqrt{x^2+y^2+(z-z_o)^2}}$$

Next we recognize that the surface-charge $\sigma(\vec{r})$ must produce the negative of the above potential in the metal located at $z<0$, so that the net potential in the metal is zero. Also, because of symmetry of the situation, the potential at $z>0$ and $z<0$ should be the same. We conclude that the surface charge density $\sigma(\vec{r})$ must produce the following potential throughout the entire space:

$$\psi(\vec{r}) = -\frac{q_o}{4\pi\epsilon_o\sqrt{x^2+y^2+(-|z|-z_o)^2}} \Rightarrow \psi(\vec{k}) = -\frac{q_o}{4\pi\epsilon_o}\int_{-\infty}^{\infty}\frac{e^{-i\vec{k}\cdot\vec{r}}}{\sqrt{x^2+y^2+(|z|+z_o)^2}}\, d\vec{r}$$

$$= -\frac{q_o}{2\pi\epsilon_o}\int_{z=0}^{\infty}\int_x\int_y \frac{\cos(k_z z)\, e^{-i\vec{k}_\parallel\cdot\vec{r}_\parallel}}{\sqrt{r_\parallel^2+(z+z_o)^2}}\, d\vec{r}_\parallel\, dz = -\frac{q_o}{\epsilon_o}\int_{z=0}^{\infty}\int_{r_\parallel=0}^{\infty}\frac{r_\parallel\cos(k_z z)\, J_o(k_\parallel r_\parallel)}{\sqrt{r_\parallel^2+(z+z_o)^2}}\, dr_\parallel\, dz \Rightarrow$$

$$\Psi(\vec{k}) = -\frac{q_0}{\epsilon_0}\int_{z=0}^{\infty} \cos(k_z z)\,\frac{e^{-k_\parallel (z+z_0)}}{k_\parallel}\,dz = -\frac{q_0 e^{-k_\parallel z_0}}{\epsilon_0 k_\parallel}\int_0^{\infty} e^{-k_\parallel z}\cos(k_z z)\,dz$$

G.R. 6.554-1

G.R. 3.893-2

$$= -\frac{q_0 e^{-k_\parallel z_0}}{\epsilon_0 k_\parallel}\,\frac{k_\parallel}{k_\parallel^2+k_z^2} = -\frac{q_0 e^{-k_\parallel z_0}}{\epsilon_0 k^2}$$

Having found $\Psi(\vec{k})$, it's now easy to find $\rho(\vec{k})$, because $\Psi(\vec{k}) = \frac{\rho(\vec{k})}{\epsilon_0 k^2}$

$$\Rightarrow \rho(\vec{k}) = \epsilon_0 k^2 \Psi(\vec{k}) = -q_0 e^{-k_\parallel z_0}$$

The final step is to inverse Fourier Transform $\rho(\vec{k})$ to find $\rho(\vec{r})$:

$$\rho(\vec{r}) = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} \rho(\vec{k})\,e^{+i\vec{k}\cdot\vec{r}}\,d\vec{k} = -\frac{q_0}{(2\pi)^3}\int_{-\infty}^{\infty} e^{-k_\parallel z_0}\,e^{i\vec{k}_\parallel\cdot\vec{r}_\parallel}\,e^{ik_z z}\,d\vec{k}_\parallel\,dk_z$$

$$= -\frac{q_0}{(2\pi)^2}\,\delta(z)\int_{k_\parallel=0}^{\infty}\int_{\theta=0}^{2\pi} e^{-k_\parallel z_0}\,e^{ik_\parallel r_\parallel\cos\theta}\,k_\parallel\,dk_\parallel\,d\theta$$

$$= -\frac{q_0}{2\pi}\,\delta(z)\int_0^{\infty} k_\parallel e^{-k_\parallel z_0} J_0(k_\parallel r_\parallel)\,dk_\parallel = -\frac{q_0}{2\pi}\,\delta(z)\,\frac{z_0}{(r_\parallel^2+z_0^2)^{3/2}}$$

The relation between $\rho(\vec{r})$ and $\sigma(\vec{r}_\parallel)$, the surface charge density, is $\rho(\vec{r}) = \sigma(\vec{r}_\parallel)\,\delta(z)$. We thus have:

$$\sigma(\vec{r}_\parallel) = -\frac{q_0 z_0}{2\pi(x^2+y^2+z_0^2)^{3/2}}$$

This is precisely the result that one will get from the traditional method of images, which involves replacing the perfect conductor with a charge $-q_0$ located at $(x,y,z) = (0,0,-z_0)$.

The perpendicular component of the E-field at the surface produced by each charge is $-\frac{q_0}{4\pi\epsilon_0}\frac{z_0}{(x^2+y^2+z_0^2)^{3/2}}$. The total $\vec{D}_\perp$ then gives $\sigma(\vec{r}_\parallel)$ at the surface.