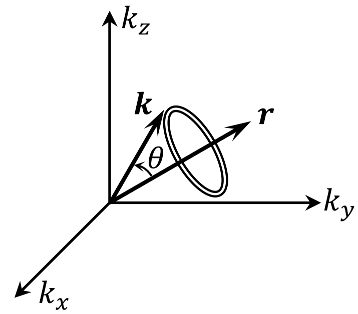
Problem 4.5) a) In the Fourier domain, the scalar potential $\psi(\boldsymbol{k})$ is readily obtained from the charge-density distribution $\rho(\boldsymbol{k})$, as follows:



$$\rho(\boldsymbol{k}) = 4\pi\rho_0[\sin(kR) - kR\cos(kR)]/k^3. \tag{1}$$

$$\psi(\boldsymbol{k}) = \frac{\rho(k)}{\varepsilon_0 k^2} = (4\pi\rho_0/\varepsilon_0)\,[\sin(kR) - kR\cos(kR)]/k^5. \tag{2}$$

An inverse Fourier transformation now yields

$$\begin{aligned}\psi(\boldsymbol{r}) &= (2\pi)^{-3}\int_{-\infty}^{\infty}\psi(\boldsymbol{k})\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})d\boldsymbol{k}\\ &= \frac{4\pi\rho_0}{(2\pi)^3\varepsilon_0}\int_{k=0}^{\infty}\int_{\theta=0}^{\pi}\{[\sin(kR) - kR\cos(kR)]/k^5\}\exp(\mathrm{i}kr\cos\theta)2\pi k^2\sin\theta\, dkd\theta\\ &= \frac{\rho_0}{\pi\varepsilon_0}\int_{k=0}^{\infty}\{[\sin(kR) - kR\cos(kR)]/k^3\}\left[\int_{\theta=0}^{\pi}\sin\theta\exp(\mathrm{i}kr\cos\theta)d\theta\right]dk. \end{aligned} \tag{3}$$

The elementary integral over θ is evaluated below:

$$\int_{\theta=0}^{\pi}\sin\theta\exp(\mathrm{i}kr\cos\theta)\,d\theta = \int_{-1}^{+1}\exp(\mathrm{i}krx)\,dx = \frac{\exp(\mathrm{i}kr) - \exp(-\mathrm{i}kr)}{\mathrm{i}kr} = \frac{2\sin(kr)}{kr}. \tag{3a}$$

Subsequently, noting that $\frac{d}{dk}[\sin(kR)/k] = [kR\cos(kR) - \sin(kR)]/k^2$, Eq.(3) is rewritten as

$$\psi(\boldsymbol{r}) = -\frac{2\rho_0}{\pi\varepsilon_0 r}\int_0^{\infty}\left\{\frac{d}{dk}[\sin(kR)/k]\right\}[\sin(kr)/k^2]dk. \tag{4}$$

Let the function $f(r)$ be the integral appearing in Eq.(4), namely,

$$f(r) = \int_0^{\infty}\left\{\frac{d}{dk}[\sin(kR)/k]\right\}[\sin(kr)/k^2]dk. \tag{5}$$

Differentiating $f(r)$ with respect to r twice, we find

$$f''(r) = -\int_0^{\infty}\left\{\frac{d}{dk}[\sin(kR)/k]\right\}\sin(kr)\,dk. \tag{6}$$

The function $f''(r)$ may be evaluated using the method of integration by parts, as follows:

$$\begin{aligned} f''(r) &= -\underbrace{[\sin(kR)/k]\sin(kr)|_{k=0}^{\infty}}_{\to 0} + \int_0^{\infty}[\sin(kR)/k][r\cos(kr)]dk\\ &= r\int_0^{\infty}\frac{\sin[k(R+r)] + \sin[k(R-r)]}{2k}dk\\ &= ½r\left\{\int_0^{\infty}\{\sin[k(R+r)]/k\}dk + \int_0^{\infty}\{\sin[k(R-r)]/k\}dk\right\}\\ &= ½r\left[\int_0^{\infty}(\sin x/x)dx \underset{r\lessgtr R}{\pm} \int_0^{\infty}(\sin x/x)dx\right] = \begin{cases} ½\pi r; & r < R,\\ 0; & r > R.\end{cases}\end{aligned} \tag{7}$$

$$\boxed{\int_0^{\infty}\left(\frac{\sin x}{x}\right)dx = ½\pi}$$

The function $f''(r)$ should now be twice integrated to yield $f(r)$, that is,

$$f(r) = \begin{cases} \alpha + \beta r + (\pi r^3/12); & r < R, \\ \gamma + \delta r; & r > R. \end{cases} \tag{8}$$

Here $\alpha, \beta, \gamma, \delta$ are constants of integration, to be determined shortly. From Eq.(4) we have

$$\psi(\boldsymbol{r}) = -\frac{2\rho_0}{\pi\varepsilon_0 r} f(r) = -\frac{2\rho_0}{\pi\varepsilon_0} \begin{cases} (\alpha/r) + \beta + (\pi r^2/12); & r < R, \\ (\gamma/r) + \delta; & r > R. \end{cases} \tag{9}$$

The potential at $r = \infty$ must vanish; therefore, $\delta = 0$. Also, at the sphere center, $r = 0$, the potential must be finite, which implies that $\alpha = 0$. Furthermore, at the sphere surface, $r = R$, the potential must be continuous; thus $\beta + (\pi R^2/12) = \gamma/R$. Therefore, $\gamma = [\beta + (\pi R^2/12)]R$. We will have

$$\psi(\boldsymbol{r}) = -\frac{2\rho_0}{\pi\varepsilon_0} \begin{cases} \beta + (\pi r^2/12); & r < R, \\ [\beta + (\pi R^2/12)]R/r\,; & r > R. \end{cases} \tag{10}$$

The E-field is obtained by evaluating the gradient of $\psi(\boldsymbol{r})$, namely,

$$\boldsymbol{E}(\boldsymbol{r}) = -\boldsymbol{\nabla}\psi(\boldsymbol{r}) = -[\partial\psi(\boldsymbol{r})/\partial r]\hat{\boldsymbol{r}} = \frac{2\rho_0\hat{\boldsymbol{r}}}{\pi\varepsilon_0} \begin{cases} \pi r/6\,; & r < R, \\ -[\beta + (\pi R^2/12)]R/r^2\,; & r > R. \end{cases} \tag{11}$$

Now, $\boldsymbol{E}(\boldsymbol{r})$ must also be continuous at $r = R$. Therefore, $\pi R/6 = -[\beta + (\pi R^2/12)]/R$, which yields $\beta = -\pi R^2/4$. We will finally have

$$\psi(\boldsymbol{r}) = (\rho_0/\varepsilon_0) \begin{cases} (R^2/2) - (r^2/6) \\ R^3/(3r) \end{cases} = \frac{(4\pi R^3/3)\rho_0}{4\pi\varepsilon_0} \begin{cases} (3R^2 - r^2)/(2R^3)\,; & r < R \\ 1/r\,; & r > R. \end{cases} \tag{12}$$

---

b) In the Fourier domain, the charge-density $\rho(\boldsymbol{k})$ and the scalar potential $\psi(\boldsymbol{k})$ are given by

$$\rho(\boldsymbol{k}) = 4\pi R\sigma_0 \sin(kR)/k. \tag{13}$$

$$\psi(\boldsymbol{k}) = \rho(\boldsymbol{k})/(\varepsilon_0 k^2) = 4\pi R\sigma_0 \sin(kR)/(\varepsilon_0 k^3). \tag{14}$$

An inverse Fourier transformation yields

$$\begin{aligned}\psi(\boldsymbol{r}) &= \frac{4\pi R\sigma_0}{(2\pi)^3\varepsilon_0} \int_{k=0}^{\infty} \int_{\theta=0}^{\pi} [\sin(kR)/k^3] \exp(\mathrm{i}kr\cos\theta)\, 2\pi k^2 \sin\theta\, dk d\theta \\ &= \frac{R\sigma_0}{\pi\varepsilon_0} \int_{k=0}^{\infty} [\sin(kR)/k] \Big[\underbrace{\int_{\theta=0}^{\pi} \sin\theta \exp(\mathrm{i}kr\cos\theta) d\theta}_{2\sin(kr)/(kr);\text{ see Eq.(3a)}}\Big] dk \\ &= \frac{2R\sigma_0}{\pi\varepsilon_0 r} \int_0^{\infty} [\sin(kR)\sin(kr)/k^2] dk\end{aligned}$$

Integration by parts

$$= \frac{2R\sigma_0}{\pi\varepsilon_0 r} \left\{ \cancelto{0}{-\sin(kR)\sin(kr)/k\big|_{k=0}^{\infty}} + \int_0^{\infty} \frac{R\cos(kR)\sin(kr) + r\sin(kR)\cos(kr)}{k} dk \right\}. \tag{15}$$

Noting that $R = ½(R + r) + ½(R - r)$ and $r = ½(R + r) - ½(R - r)$, we now write

$$\psi(\boldsymbol{r}) = \frac{R\sigma_0}{\pi\varepsilon_0 r}\left\{\int_0^\infty \frac{(R+r)[\cos(kR)\sin(kr)+\sin(kR)\cos(kr)]}{k}dk + \int_0^\infty \frac{(R-r)[\cos(kR)\sin(kr)-\sin(kR)\cos(kr)]}{k}dk\right\}$$

$$= \frac{R\sigma_0}{\pi\varepsilon_0 r}\left\{(R+r)\int_0^\infty \frac{\sin[k(R+r)]}{k}dk + (R-r)\int_0^\infty \frac{\sin[k(r-R)]}{k}dk\right\}$$

$$= \frac{R\sigma_0}{\pi\varepsilon_0 r}\left[(R+r)\int_0^\infty (\sin x/x)dx \overset{\boxed{r \gtrless R}}{\pm} (R-r)\int_0^\infty (\sin x/x)dx\right] \overset{\boxed{\int_0^\infty \left(\frac{\sin x}{x}\right)dx = \frac{1}{2}\pi}}{=} \frac{R\sigma_0}{2\varepsilon_0 r}[(R+r) \pm (R-r)].$$

Finally, in terms of the total charge $Q = 4\pi R^2\sigma_0$, the scalar potential of the shell (both inside and outside) is given by

$$\psi(\boldsymbol{r}) = \frac{4\pi R^2\sigma_0}{4\pi\varepsilon_0}\begin{cases} 1/R\,; & \text{(inside the shell, } r \le R\text{),} \\ 1/r\,; & \text{(outside the shell, } r \ge R\text{).} \end{cases} \tag{16}$$