Problem 4)

$$\rho(\vec{r}) = q\delta(\vec{r}) \Rightarrow \rho(\vec{k}) = \int_{-\infty}^{\infty} \rho(\vec{r})\, e^{-i\vec{k}\cdot\vec{r}}\, d\vec{r} = \int_{-\infty}^{\infty} q\delta(\vec{r})\, e^{-i\vec{k}\cdot\vec{r}}\, d\vec{r}$$

$$\Rightarrow \rho(\vec{k}) = q$$

Maxwell's first equation: $\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0 \Rightarrow \vec{\nabla}\cdot\int \vec{E}(\vec{k})\, e^{+i\vec{k}\cdot\vec{r}}\, d\vec{k} = \frac{1}{\epsilon_0}\int_{-\infty} \rho(\vec{k})\, e^{+i\vec{k}\cdot\vec{r}}\, d\vec{k}$

$$\Rightarrow \int_{-\infty}^{\infty} i\vec{k}\cdot\vec{E}(\vec{k})\, e^{i\vec{k}\cdot\vec{r}}\, d\vec{k} = \frac{1}{\epsilon_0}\int_{-\infty}^{\infty} \rho(\vec{k})\, e^{+i\vec{k}\cdot\vec{r}}\, d\vec{k} \Rightarrow i\vec{k}\cdot\vec{E}(\vec{k}) = \frac{1}{\epsilon_0}\rho(\vec{k}) = q/\epsilon_0$$

In electrostatic problems $\vec{\nabla}\times\vec{E}(\vec{r}) = 0 \Rightarrow \vec{k}\times\vec{E}(\vec{k}) = 0 \Rightarrow \vec{E}(\vec{k})$ is purely longitudinal, i.e., it does not have a transverse component $\perp$ to $\vec{k}$. Thus $\vec{E}(\vec{k})$ is parallel to $\vec{k}$. From above we find $\vec{E}(\vec{k}) = -\frac{iq}{\epsilon_0}\frac{\hat{k}}{k}$.

In problem 4(d) we saw that $\mathcal{F}\{\hat{r}/r^2\} = -\frac{4\pi i\hat{k}}{k}$. Therefore:

$$\vec{E}(\vec{r}) = \frac{q}{4\pi\epsilon_0}\left(\frac{\hat{r}}{r^2}\right)$$