Problem 3)

Maxwell's first equation: $\vec{\nabla}\cdot\vec{D}=\rho_{free} \Rightarrow \epsilon_0\vec{\nabla}\cdot\vec{E}=\rho_{total}$ where $\rho_{total}=\rho_{free}-\vec{\nabla}\cdot\vec{P}$.

Maxwell's 2nd equation: $\vec{\nabla}\times\vec{H}=\vec{J}_{free}+\frac{\partial\vec{D}}{\partial t} \Rightarrow \vec{\nabla}\times\vec{B}=\mu_0\vec{J}_{total}+\frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}$ where

$$\vec{J}_{total}=\vec{J}_{free}+\frac{\partial\vec{P}}{\partial t}+\frac{1}{\mu_0}\vec{\nabla}\times\vec{M} \quad\text{and}\quad \frac{1}{c^2}=\mu_0\epsilon_0.$$

Maxwell's 3rd equation: $\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}=-\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{A}) \Rightarrow \vec{\nabla}\times(\vec{E}+\frac{\partial\vec{A}}{\partial t})=0$

$$\Rightarrow \vec{E}+\frac{\partial\vec{A}}{\partial t}=-\vec{\nabla}\psi \Rightarrow \vec{E}=-\vec{\nabla}\psi-\frac{\partial\vec{A}}{\partial t}.$$

Maxwell's 4th equation: $\vec{\nabla}\cdot\vec{B}=0 \Rightarrow \vec{B}=\vec{\nabla}\times\vec{A}$.

Assuming $\rho_{total}(\vec{r},t)=\rho_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}$, $\vec{J}_{total}(\vec{r},t)=\vec{J}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}$, etc., we can write $\vec{A}(\vec{r},t)=(\vec{A}_\parallel+\vec{A}_\perp)e^{i(\vec{k}\cdot\vec{r}-\omega t)}$ and $\psi(\vec{r},t)=\psi_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}$.

✓ $\vec{B}=\vec{\nabla}\times\vec{A} \Rightarrow (\vec{B}_\parallel+\vec{B}_\perp)e^{i(\vec{k}\cdot\vec{r}-\omega t)}=i\vec{k}\times(\vec{A}_\parallel+\vec{A}_\perp)e^{i(\vec{k}\cdot\vec{r}-\omega t)} \Rightarrow \begin{cases}\vec{B}_\parallel=0\\ \vec{B}_\perp=i\vec{k}\times\vec{A}_\perp\end{cases}$

✓ Coulomb gauge: $\vec{\nabla}\cdot\vec{A}(\vec{r},t)=0 \Rightarrow i\vec{k}\cdot(\vec{A}_\parallel+\vec{A}_\perp)e^{i(\vec{k}\cdot\vec{r}-\omega t)}=0 \Rightarrow \vec{A}_\parallel=0.$

✓ $\vec{E}(\vec{r},t)=-\vec{\nabla}\psi(\vec{r},t)-\frac{\partial\vec{A}(\vec{r},t)}{\partial t} \Rightarrow (\vec{E}_\parallel+\vec{E}_\perp)=-i\psi_0\vec{k}+i\omega\vec{A}_\perp \Rightarrow \begin{cases}\vec{E}_\parallel=-i\psi_0\vec{k}\\ \vec{E}_\perp=i\omega\vec{A}_\perp\end{cases}$

✓ From 1st equation: $\epsilon_0\vec{\nabla}\cdot\vec{E}=\rho_{total} \Rightarrow i\epsilon_0\vec{k}\cdot(\vec{E}_\parallel+\vec{E}_\perp)=\rho_0 \Rightarrow$

$$i\epsilon_0 k E_\parallel=\rho_0 \Rightarrow i\epsilon_0 k(-i\psi_0 k)=\rho_0 \Rightarrow \psi_0=\frac{\rho_0}{\epsilon_0 k^2}$$

✓ From Maxwell's 2nd equation: $\vec{\nabla}\times\vec{B}=\mu_0\vec{J}_{total}+\frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} \Rightarrow i\vec{k}\times\vec{B}=\mu_0\vec{J}_0-i\frac{\omega}{c^2}\vec{E}$

$$\Rightarrow i\vec{k}\times(i\vec{k}\times\vec{A}_\perp)=\mu_0\vec{J}_0-\frac{i\omega}{c^2}(-i\psi_0\vec{k}+i\omega\vec{A}_\perp) \Rightarrow k^2\vec{A}_\perp=\mu_0\vec{J}_{0\parallel}+\mu_0\vec{J}_{0\perp}-\frac{\omega}{c^2}\psi_0\vec{k}$$

$$+\frac{\omega^2}{c^2}\vec{A}_\perp \Rightarrow \begin{cases}(k^2-\omega^2/c^2)\vec{A}_\perp=\mu_0\vec{J}_{0\perp}\\ \frac{\omega}{c^2}\psi_0\vec{k}=\mu_0\vec{J}_{0\parallel}\end{cases} \Rightarrow \begin{cases}\vec{A}_\perp=\mu_0\vec{J}_{0\perp}/(k^2-\omega^2/c^2)\\ \omega\rho_0=kJ_{0\parallel}\end{cases}$$

← This is just $\vec{\nabla}\cdot\vec{J}_{tot}+\frac{\partial\rho_{tot}}{\partial t}=0$.

Therefore, in the Coulomb gauge, $\psi_0 = \dfrac{\rho_0}{\epsilon_0 k^2}$ and $\vec{A}_\perp = \dfrac{\mu_0 \vec{J}_{0\perp}}{k^2 - \omega^2/c^2}$, which yield

→ $$\vec{B} = \vec{B}_\perp = i\vec{k}\times\vec{A}_\perp = \frac{i\mu_0 \vec{k}\times\vec{J}_{0\perp}}{k^2-\omega^2/c^2} = \frac{i\mu_0 \vec{k}\times\vec{J}_0}{k^2-\omega^2/c^2}$$ ← Same as Lorentz gauge; See Prob. 16 ✓

$$\vec{E} = \vec{E}_\parallel + \vec{E}_\perp = -i\psi_0\vec{k} + i\omega\vec{A}_\perp = -i\frac{\rho_0\vec{k}}{\epsilon_0 k^2} + \frac{i\mu_0\omega\vec{J}_{0\perp}}{k^2-\omega^2/c^2}.$$

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→ Now, in the Lorentz gauge (See problem 16) we have:

$\psi_0 = \dfrac{\rho_0}{\epsilon_0(k^2-\omega^2/c^2)}$ and $\vec{A}_0 = \dfrac{\mu_0\vec{J}_0}{k^2-\omega^2/c^2}$. Also, Continuity equation $\vec{k}\cdot\vec{J}_0 = \omega\rho_0$

Therefore, $$\vec{E} = -i\vec{k}\psi_0 + i\omega\vec{A}_0 = \frac{-i(\rho_0/\epsilon_0)\vec{k} + i\mu_0\omega\vec{J}_0}{k^2-\omega^2/c^2}$$

$$= \frac{-i(\rho_0/\epsilon_0)\vec{k} + i\mu_0\omega\vec{J}_{0\parallel} + i\mu_0\omega\vec{J}_{0\perp}}{k^2-\omega^2/c^2} = \frac{-i(\rho_0/\epsilon_0)\vec{k} + i\mu_0\omega^2\rho_0\vec{k}/k^2 + i\mu_0\omega\vec{J}_{0\perp}}{k^2-\omega^2/c^2}$$

$$= -i\frac{\rho_0\vec{k}}{\epsilon_0 k^2}\,\underbrace{\cancel{\frac{k^2-\omega^2/c^2}{k^2-\omega^2/c^2}}}_{1} + \frac{i\mu_0\omega\vec{J}_{0\perp}}{k^2-\omega^2/c^2}$$ ← Same as $\vec{E}$ in Coulomb gauge.

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