Problem 1)

Let $\vec{B}(\vec{r},t)=\vec{\nabla}\times\vec{A}(\vec{r},t)$ where $\vec{A}(\vec{r},t)$ is the vector potential field. In the Fourier domain $\vec{B}(\vec{k},\omega)=i\vec{k}\times\vec{A}(\vec{k},\omega)$. This is all one needs to prove that Maxwell's 4th equation is automatically satisfied:

$$\vec{\nabla}\cdot\vec{B}(\vec{r},t)=0 \Rightarrow i\vec{k}\cdot\vec{B}(\vec{k},\omega)=0 \Rightarrow i^2\vec{k}\cdot[\vec{k}\times\vec{A}(\vec{k},\omega)]=0 \Rightarrow (\vec{k}\times\vec{k})\cdot\vec{A}(\vec{k},\omega)=0 \checkmark$$

Maxwell's 3rd equation: $\vec{\nabla}\times\vec{E}(\vec{r},t)=-\frac{\partial}{\partial t}\vec{B}(\vec{r},t) \Rightarrow \vec{\nabla}\times(\vec{E}+\frac{\partial\vec{A}}{\partial t})=0 \Rightarrow$

$\vec{E}+\frac{\partial\vec{A}}{\partial t}=-\vec{\nabla}\psi(\vec{r},t)$, where $\psi(\vec{r},t)$ is the scalar potential field. $\Rightarrow$

$$\vec{E}(\vec{r},t)=-\vec{\nabla}\psi(\vec{r},t)-\frac{\partial}{\partial t}\vec{A}(\vec{r},t) \Rightarrow \vec{E}(\vec{k},\omega)=-i\vec{k}\psi(\vec{k},\omega)+i\omega\vec{A}(\vec{k},\omega)$$

Maxwell's 1st equation: $\vec{\nabla}\cdot\vec{D}(\vec{r},t)=\rho_{free}(\vec{r},t) \Rightarrow i\vec{k}\cdot[\epsilon_0\vec{E}(\vec{k},\omega)+\vec{P}(\vec{k},\omega)]=\rho_{free}(\vec{k},\omega)$

$$\Rightarrow \epsilon_0 k^2\psi(\vec{k},\omega)-\epsilon_0\omega\vec{k}\cdot\vec{A}(\vec{k},\omega)+i\vec{k}\cdot\vec{P}(\vec{k},\omega)=\rho_{free}(\vec{k},\omega).$$

So far we have only specified the transverse component of $\vec{A}(\vec{k},\omega)$; its longitudinal component is as yet undetermined. The Lorentz gauge, for example, can fix the longitudinal component of $\vec{A}$.

Lorentz gauge: $\vec{\nabla}\cdot\vec{A}(\vec{r},t)+\frac{1}{c^2}\frac{\partial\psi(\vec{r},t)}{\partial t}=0 \Rightarrow i\vec{k}\cdot\vec{A}(\vec{k},\omega)-\frac{i\omega}{c^2}\psi(\vec{k},\omega)=0 \Rightarrow$

$\vec{k}\cdot\vec{A}(\vec{k},\omega)=\frac{\omega}{c^2}\psi(\vec{k},\omega)$ ← Lorentz condition (or gauge)

Maxwell's 1st equation thus becomes: $\epsilon_0(k^2-\frac{\omega^2}{c^2})\psi(\vec{k},\omega)=\rho_{free}(\vec{k},\omega)-i\vec{k}\cdot\vec{P}(\vec{k},\omega)$

$$\Rightarrow \psi(\vec{k},\omega)=\frac{\rho_{free}(\vec{k},\omega)-i\vec{k}\cdot\vec{P}(\vec{k},\omega)}{\epsilon_0(k^2-\frac{\omega^2}{c^2})}$$

Maxwell's 2nd equation: $\vec{\nabla}\times\vec{H}=\vec{J}_{free}+\frac{\partial\vec{D}}{\partial t} \Rightarrow \vec{\nabla}\times(\mu_0\vec{H}+\vec{M})=\mu_0\vec{J}_{free}+\vec{\nabla}\times\vec{M}+\mu_0\frac{\partial\vec{P}}{\partial t}+\mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$

$$\Rightarrow i\vec{k}\times\vec{B}(\vec{k},\omega)=\mu_0\vec{J}_{free}(\vec{k},\omega)+i\vec{k}\times\vec{M}(\vec{k},\omega)-i\omega\mu_0\vec{P}(\vec{k},\omega)-\frac{i\omega}{c^2}\vec{E}(\vec{k},\omega)$$

$$\Rightarrow i^2\vec{k}\times[\vec{k}\times\vec{A}(\vec{k},\omega)]=\mu_0\vec{J}_{free}(\vec{k},\omega)+i\vec{k}\times\vec{M}(\vec{k},\omega)-i\omega\mu_0\vec{P}(\vec{k},\omega)-\frac{\omega}{c^2}\vec{k}\psi(\vec{k},\omega)+\frac{\omega^2}{c^2}\vec{A}(\vec{k},\omega)$$

$$\Rightarrow k^2\vec{A}(\vec{k},\omega)-\cancel{[\vec{k}\cdot\vec{A}(\vec{k},\omega)]\vec{k}}=\mu_0\vec{J}_{free}(\vec{k},\omega)-i\omega\mu_0\vec{P}(\vec{k},\omega)+i\vec{k}\times\vec{M}(\vec{k},\omega)-\cancel{\frac{\omega}{c^2}\vec{k}\psi(\vec{k},\omega)}+\frac{\omega^2}{c^2}\vec{A}(\vec{k},\omega)$$

Lorentz gauge

$$\Rightarrow \vec{A}(\vec{k},\omega) = \mu_0 \frac{\vec{J}_{free}(\vec{k},\omega) - i\omega\vec{P}(\vec{k},\omega) + i\mu_0^{-1}\vec{k}\times\vec{M}(\vec{k},\omega)}{k^2-\omega^2/c^2}$$

In this way one can determine the scalar potential from the free charge distribution ρ_{free} and the bound charge distribution $-i\vec{k}\cdot\vec{P}(\vec{k},\omega)$. Similarly, the vector potential is determined from the free current distribution $\vec{J}_{free}(\vec{k},\omega)$ and the bound current distribution $-i\omega\vec{P}(\vec{k},\omega)+i\mu_0^{-1}\vec{k}\times\vec{M}(\vec{k},\omega)$. Once the potentials are found, the $\vec{E}$ and $\vec{H}$ fields can be derived from them in accordance with the preceding equations.

Verifying the Lorentz gauge: $\vec{k}\cdot\vec{A}(\vec{k},\omega) = \frac{\omega}{c^2}\psi(\vec{k},\omega) \Rightarrow$

$$\mu_0\vec{k}\cdot\vec{J}_{free}(\vec{k},\omega) - i\mu_0\omega\cancel{\vec{k}\cdot\vec{P}(\vec{k},\omega)} + i\vec{k}\cdot[\vec{k}\times\vec{M}(\vec{k},\omega)] = \frac{\omega}{\epsilon_0 c^2}\rho_{free}(\vec{k},\omega) - i\frac{\omega}{\epsilon_0 c^2}\cancel{\vec{k}\cdot\vec{P}(\vec{k},\omega)}$$

Using the relation $c=\frac{1}{\sqrt{\mu_0\epsilon_0}}$, the above equation may be written as follows:

$$\mu_0\left[\vec{k}\cdot\vec{J}_{free}(\vec{k},\omega) - \omega\rho_{free}(\vec{k},\omega)\right] + i\,\overset{\to 0}{\cancel{(\vec{k}\times\vec{k})}}\cdot\vec{M}(\vec{k},\omega) = 0 \Rightarrow i\vec{k}\cdot\vec{J}_{free}(\vec{k},\omega) - i\omega\rho_{free}(\vec{k},\omega) = 0$$

The last equation is the continuity equation $\vec{\nabla}\cdot\vec{J}_{free}(\vec{r},t)+\frac{\partial}{\partial t}\rho_{free}(\vec{r},t)$ in the Fourier domain.
