Problem 3.32) The function $f(t) = \cos(\omega_0 t)$ must first be multiplied by $\exp(-\alpha|t|)$ in order to prevent its Fourier integral from diverging as $t \to \pm\infty$. Eventually, however, we must let $\alpha \to 0$, to lift this artificial restriction on the magnitude of $f(t)$.

$$
\begin{aligned}
F(\omega) &= \lim_{\alpha\to 0} \int_{-\infty}^{\infty} \cos(\omega_0 t) \exp(-\alpha|t|) \exp(i\omega t)\, dt \\
&= \lim_{\alpha\to 0} \int_{-\infty}^{\infty} ½[\exp(i\omega_0 t) + \exp(-i\omega_0 t)] \exp(-\alpha|t|) \exp(i\omega t)\, dt \\
&= ½ \lim_{\alpha\to 0} \Big\{ \int_{-\infty}^{0} [\exp(i\omega_0 t) + \exp(-i\omega_0 t)] \exp(\alpha t) \exp(i\omega t)\, dt \\
&\qquad + \int_{0}^{\infty} [\exp(i\omega_0 t) + \exp(-i\omega_0 t)] \exp(-\alpha t) \exp(i\omega t)\, dt \Big\} \\
&= ½ \lim_{\alpha\to 0} \Big\{ \int_{-\infty}^{0} \exp\{[\alpha + i(\omega+\omega_0)]t\} dt + \int_{-\infty}^{0} \exp\{[\alpha + i(\omega-\omega_0)]t\}\, dt \\
&\qquad + \int_{0}^{\infty} \exp\{-[\alpha - i(\omega+\omega_0)]t\}\, dt + \int_{0}^{\infty} \exp\{-[\alpha - i(\omega-\omega_0)]t\}\, dt \Big\} \\
&= \lim_{\alpha\to 0} \left[\frac{½}{\alpha + i(\omega+\omega_0)} + \frac{½}{\alpha + i(\omega-\omega_0)} + \frac{½}{\alpha - i(\omega+\omega_0)} + \frac{½}{\alpha - i(\omega-\omega_0)} \right] \\
&= \lim_{\alpha\to 0} \left[\frac{½}{\alpha + i(\omega+\omega_0)} + \frac{½}{\alpha - i(\omega+\omega_0)} + \frac{½}{\alpha + i(\omega-\omega_0)} + \frac{½}{\alpha - i(\omega-\omega_0)} \right] \\
&= \lim_{\alpha\to 0} \left[\frac{\alpha}{\alpha^2 + (\omega+\omega_0)^2} + \frac{\alpha}{\alpha^2 + (\omega-\omega_0)^2} \right].
\end{aligned}
$$

The area under each of the two functions in the preceding expression is equal to π, as may be readily verified:

$$
\int_{-\infty}^{\infty} \frac{\alpha}{\alpha^2 + (\omega \pm \omega_0)^2}\, d\omega = \int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-½\pi}^{½\pi} \left(\frac{1+\tan^2\theta}{1+\tan^2\theta} \right) d\theta = \pi.
$$

In the limit when $\alpha \to 0$, the first of the above functions approaches $\pi\delta(\omega+\omega_0)$ while the second one approaches $\pi\delta(\omega-\omega_0)$. Consequently, $F(\omega) = \pi\delta(\omega+\omega_0) + \pi\delta(\omega-\omega_0)$.