Problem 3.31)

a) $\boldsymbol{M}(\boldsymbol{r},t) = M_0\hat{\boldsymbol{z}}\,[\mathrm{Rect}(x/L_x)\mathrm{Rect}(y/L_y) - \mathrm{Circ}(r_\parallel/R)]\mathrm{Rect}(z/L_z).$ ← $r_\parallel = \sqrt{x^2+y^2}$

b) $$\rho_{\text{bound}}^{(m)}(\boldsymbol{r},t) = -\boldsymbol{\nabla}\cdot\boldsymbol{M}(\boldsymbol{r},t) = -\,\partial M_z/\partial z$$
$$= M_0[\mathrm{Rect}(x/L_x)\mathrm{Rect}(y/L_y) - \mathrm{Circ}(r_\parallel/R)][\delta(z-½L_z) - \delta(z+½L_z)].$$

$$\boldsymbol{J}_{\text{bound}}^{(e)}(\boldsymbol{r},t) = \mu_0^{-1}\boldsymbol{\nabla}\times\boldsymbol{M}(\boldsymbol{r},t)$$
$$= \mu_0^{-1}M_0\{\boldsymbol{\nabla}\times[\mathrm{Rect}(x/L_x)\mathrm{Rect}(y/L_y)\mathrm{Rect}(z/L_z)\hat{\boldsymbol{z}}]$$ ← Cartesian coordinates
$$-\boldsymbol{\nabla}\times[\mathrm{Circ}(r_\parallel/R)\mathrm{Rect}(z/L_z)\hat{\boldsymbol{z}}]\}$$ ← Cylindrical coordinates
$$= \mu_0^{-1}M_0\mathrm{Rect}(x/L_x)[\delta(y+½L_y) - \delta(y-½L_y)]\mathrm{Rect}(z/L_z)\hat{\boldsymbol{x}}$$
$$-\mu_0^{-1}M_0[\delta(x+½L_x) - \delta(x-½L_x)]\mathrm{Rect}(y/L_y)\mathrm{Rect}(z/L_z)\hat{\boldsymbol{y}}$$
$$-\mu_0^{-1}M_0\delta(r_\parallel - R)\mathrm{Rect}(z/L_z)\hat{\boldsymbol{\phi}}.$$

c) The Fourier transform of the bound charge-density distribution is evaluated as follows:

$$\rho_{\text{bound}}^{(m)}(\boldsymbol{k},\omega) = \int_{-\infty}^{\infty}\rho_{\text{bound}}^{(m)}(\boldsymbol{r},t)\exp[-\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\,\mathrm{d}\boldsymbol{r}\mathrm{d}t$$
$$= 2\pi M_0\delta(\omega)\{\int_{-\infty}^{\infty}\mathrm{Rect}(x/L_x)\exp(-\mathrm{i}k_x x)\,\mathrm{d}x\int_{-\infty}^{\infty}\mathrm{Rect}(y/L_y)\exp(-\mathrm{i}k_y y)\,\mathrm{d}y$$

$\boldsymbol{k}_\parallel = k_x\hat{\boldsymbol{x}} + k_y\hat{\boldsymbol{y}}$ →
$$-\int_{r_\parallel=0}^{\infty}\mathrm{Circ}(r_\parallel/R)\int_{\phi=0}^{2\pi}\exp(-\mathrm{i}k_\parallel r_\parallel\cos\phi)\,r_\parallel\mathrm{d}\phi\mathrm{d}r_\parallel\Big\}$$ ← G&R **3.915**-2
$$\times\int_{-\infty}^{\infty}[\delta(z-½L_z) - \delta(z+½L_z)]\exp(-\mathrm{i}k_z z)\,\mathrm{d}z$$
$$= 2\pi M_0\delta(\omega)\left\{\int_{-L_x/2}^{L_x/2}\exp(-\mathrm{i}k_x x)\,\mathrm{d}x\int_{-L_y/2}^{L_y/2}\exp(-\mathrm{i}k_y y)\,\mathrm{d}y - 2\pi\int_0^R r_\parallel J_0(k_\parallel r_\parallel)\mathrm{d}r_\parallel\right\}$$
$$\times[\exp(-½\mathrm{i}L_z k_z) - \exp(½\mathrm{i}L_z k_z)]$$ G&R **5.56**-2
$$= 2\pi M_0\delta(\omega)\left\{\frac{2\sin(½L_x k_x)}{k_x}\times\frac{2\sin(½L_y k_y)}{k_y} - \frac{2\pi R}{k_\parallel}J_1(k_\parallel R)\right\}[-2\mathrm{i}\sin(½L_z k_z)]$$
$$= -\mathrm{i}4\pi M_0\delta(\omega)\left\{L_x L_y\mathrm{sinc}\left(\frac{L_x k_x}{2\pi}\right)\mathrm{sinc}\left(\frac{L_y k_y}{2\pi}\right) - 2\pi R\frac{J_1[R(k_x^2+k_y^2)^{½}]}{(k_x^2+k_y^2)^{½}}\right\}\sin(½L_z k_z).$$

Similarly, the Fourier transform of the bound current-density distribution is found to be

$$\boldsymbol{J}_{\text{bound}}^{(e)}(\boldsymbol{k},\omega) = \int_{-\infty}^{\infty}\boldsymbol{J}_{\text{bound}}^{(e)}(\boldsymbol{r},t)\exp[-\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]\,\mathrm{d}\boldsymbol{r}\mathrm{d}t$$
$$= 2\pi\mu_0^{-1}M_0\delta(\omega)\left\{\hat{\boldsymbol{x}}\int_{-L_x/2}^{L_x/2}\exp(-\mathrm{i}k_x x)\mathrm{d}x\int_{-\infty}^{\infty}[\delta(y+½L_y) - \delta(y-½L_y)]\exp(-\mathrm{i}k_y y)\mathrm{d}y\right.$$
$$-\hat{\boldsymbol{y}}\int_{-\infty}^{\infty}[\delta(x+½L_x) - \delta(x-½L_x)]\exp(-\mathrm{i}k_x x)\mathrm{d}x\int_{-L_y/2}^{L_y/2}\exp(-\mathrm{i}k_y y)\mathrm{d}y$$

$\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{z}}\times\hat{\boldsymbol{r}}_\parallel$ →
$$\left.-\hat{\boldsymbol{z}}\times\int_{r_\parallel=0}^{\infty}\int_{\phi=0}^{2\pi}\hat{\boldsymbol{r}}_\parallel\delta(r_\parallel - R)\exp(-\mathrm{i}k_\parallel r_\parallel\cos\phi)r_\parallel\mathrm{d}r_\parallel\mathrm{d}\phi\right\}\int_{-L_z/2}^{L_z/2}\exp(-\mathrm{i}k_z z)\mathrm{d}z$$
$$= 2\pi\mu_0^{-1}M_0\delta(\omega)\{2\mathrm{i}L_x\mathrm{sinc}(L_x k_x/2\pi)\sin(½L_y k_y)\hat{\boldsymbol{x}} - 2\mathrm{i}L_y\mathrm{sinc}(L_y k_y/2\pi)\sin(½L_x k_x)\hat{\boldsymbol{y}}$$

G&R **3.915**-2 → $-\hat{\boldsymbol{z}} \times \hat{\boldsymbol{k}}_\parallel \int_{r_\parallel=0}^{\infty} r_\parallel \delta(r_\parallel - R) \int_{\phi=0}^{2\pi} \cos\phi \exp(-\mathrm{i} k_\parallel r_\parallel \cos\phi) \mathrm{d}\phi \mathrm{d}r_\parallel \Big\} L_z \mathrm{sinc}(L_z k_z / 2\pi)$

$\boldsymbol{k}_\parallel \times \hat{\boldsymbol{z}}$ ←

$$= \mathrm{i}2\pi\mu_0^{-1} M_0 \delta(\omega) L_z \mathrm{sinc}(L_z k_z/2\pi) \big\{ L_x L_y \mathrm{sinc}(L_x k_x/2\pi)\, \mathrm{sinc}(L_y k_y/2\pi) \overbrace{(k_y \hat{\boldsymbol{x}} - k_x \hat{\boldsymbol{y}})}$$

$\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{z}} \times \hat{\boldsymbol{k}}_\parallel$ →

$$+ 2\pi R\, J_1[R(k_x^2 + k_y^2)^{1/2}] \hat{\boldsymbol{\phi}} \big\} .$$

← $k_\parallel = (k_x^2 + k_y^2)^{1/2}$
