Problem 3.30) a) In the limit when $\alpha \to \infty$, the symmetric function $g_\alpha(x)$ becomes tall and narrow, with an area that is always equal to 2. Therefore, $\lim_{\alpha\to\infty} g_\alpha(x) = 2\delta(x)$.

Odd terms omitted, as they integrate to zero.

b) $$\int_{-\infty}^{\infty} f(x) g_\alpha(x)\mathrm{d}x = \int_{-\infty}^{\infty} \sum_{n=0}^{\infty} f_n \frac{x^n}{n!} \alpha \exp(-\alpha|x|)\mathrm{d}x = 2\alpha \int_0^{\infty} \sum_{n=0}^{\infty} f_{2n} \frac{x^{2n}}{(2n)!} \exp(-\alpha x)\mathrm{d}x$$

$$= 2\alpha \sum_{n=0}^{\infty} \frac{f_{2n}}{(2n)!} \int_0^{\infty} x^{2n} \exp(-\alpha x)\mathrm{d}x = 2\alpha \sum_{n=0}^{\infty} \frac{f_{2n}}{\alpha^{2n+1}} = 2f_0 + 2\sum_{n=1}^{\infty} \frac{f_{2n}}{\alpha^{2n}}.$$

In the limit $\alpha \to \infty$, all the terms under the summation sign vanish, leaving $2f_0 = 2f(x=0)$ as the value of the integral. This, of course, is just a manifestation of the sifting property of $2\delta(x)$.

c) The function $h_\beta(x)$ is the derivative of $-\tfrac{1}{2}\sqrt{\beta}\exp(-\beta x^2)$, which approaches $-\tfrac{1}{2}\sqrt{\pi}\,\delta(x)$ in the limit when $\beta \to \infty$. Therefore, the limit of $h_\beta(x)$ is the delta-function-derivative $-\tfrac{1}{2}\sqrt{\pi}\,\delta'(x)$.

Even terms omitted, as they integrate to zero.

d) $$\int_{-\infty}^{\infty} f(x) h_\beta(x)\mathrm{d}x = \int_{-\infty}^{\infty} \sum_{n=0}^{\infty} f_n \frac{x^n}{n!} \beta^{3/2} x \exp(-\beta x^2)\mathrm{d}x = 2\beta^{3/2} \int_0^{\infty} \sum_{n=0}^{\infty} f_{2n+1} \frac{x^{2n+1}}{(2n+1)!} x \exp(-\beta x^2)\mathrm{d}x$$

$$= 2\beta^{3/2} \sum_{n=0}^{\infty} \frac{f_{2n+1}}{(2n+1)!} \int_0^{\infty} x^{2n+2} \exp(-\beta x^2)\mathrm{d}x = 2\beta^{3/2} \sum_{n=0}^{\infty} \frac{f_{2n+1}}{(2n+1)!} \frac{(2n+1)!!\sqrt{\pi}}{2^{n+2}\beta^{n+\frac{3}{2}}}$$

$$= \frac{1}{2}\sqrt{\pi}\, f_1 + \frac{1}{2}\sqrt{\pi} \sum_{n=1}^{\infty} \frac{f_{2n+1}}{n!\,(4\beta)^n}.$$

$$\frac{(2n+1)!!}{(2n+1)!} = \frac{1}{(2n)!!} = \frac{1}{2^n n!}$$

When $\beta \to \infty$, the terms under the summation sign vanish, leaving $\tfrac{1}{2}\sqrt{\pi} f_1 = \tfrac{1}{2}\sqrt{\pi}\,\mathrm{d}f(x)/\mathrm{d}x|_{x=0}$ as the value of the integral. This is nothing more nor less than the sifting property of $-\tfrac{1}{2}\sqrt{\pi}\,\delta'(x)$.