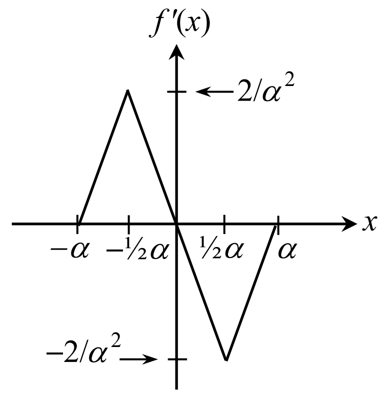
Problem 3.29) a) The symmetry of the function $f(x)$ allows us to integrate over the interval $(0,\alpha)$, then multiply the result by 2 to find the area under the function, that is,

$$\int_{-\infty}^{\infty} f(x)dx = 2\left\{\int_0^{½\alpha}(2/\alpha)[½-(x/\alpha)^2]dx + \int_{½\alpha}^{\alpha}(2/\alpha)[1-(x/\alpha)]^2dx\right\}$$
$$= 4\left[\int_0^{½}(½-y^2)dy + \int_{½}^{1}(1-y)^2dy\right]$$
$$= 4\left[(½y-⅓y^3)\big|_0^{½} - ⅓(1-y)^3\big|_{½}^{1}\right] = 4\left(\tfrac{1}{4}-\tfrac{1}{24}+\tfrac{1}{24}\right) = 1$$

Since $f(x)$ is tall, narrow, symmetric around $x=0$, and has unit area, in the limit when $\alpha\to 0$, the function $f(x)$ approaches a delta-function.

b) Differentiation of $f(x)$ with respect to x yields

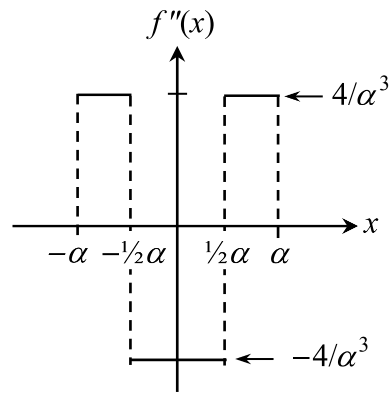
$$f'(x) = \begin{cases} 0, & x \le -\alpha; \\ (4/\alpha^2)[1+(x/\alpha)], & -\alpha \le x \le -½\alpha; \\ -4x/\alpha^3, & -½\alpha \le x \le ½\alpha; \\ -(4/\alpha^2)[1-(x/\alpha)], & ½\alpha \le x \le \alpha; \\ 0, & x \ge \alpha. \end{cases}$$



The area under each half of the function $f'(x)$ is $1/\alpha$. If the product $f'(x)g(x)$ of an arbitrary function $g(x)$ with $f'(x)$ is integrated over x, the value of the integral on the left-hand side of the x-axis will be $\alpha^{-1}g(-½\alpha)$, while that on the right-hand side will be $-\alpha^{-1}g(½\alpha)$. The total integral of $f'(x)g(x)$ will thus be $[g(-½\alpha)-g(½\alpha)]/\alpha$, which approaches $-g'(0)$ as $\alpha\to 0$. This is the expected sifting behavior of $\delta'(x)$. Therefore, in the limit when $\alpha\to 0$, the function $f'(x)$ approaches $\delta'(x)$.

c) Differentiating $f'(x)$ with respect to x, we find

$$f''(x) = \begin{cases} 0, & x \le -\alpha; \\ 4/\alpha^3, & -\alpha \le x \le -½\alpha; \\ -4/\alpha^3, & -½\alpha \le x \le ½\alpha; \\ 4/\alpha^3, & ½\alpha \le x \le \alpha; \\ 0, & x \ge \alpha. \end{cases}$$



The area under each segment of the function $f''(x)$ is $2/\alpha^2$. If the product $f''(x)g(x)$ of an arbitrary function $g(x)$ with $f''(x)$ is integrated over x, the values of the integral over the four segments of $f''(x)$ will be, from left to right, $(2/\alpha^2)g(-¾\alpha)$, $-(2/\alpha^2)g(-¼\alpha)$, $-(2/\alpha^2)g(¼\alpha)$ and $(2/\alpha^2)g(¾\alpha)$. The total integral of $f''(x)g(x)$ will thus be

$$\frac{g(-¾\alpha)-g(-¼\alpha)-g(¼\alpha)+g(¾\alpha)}{\alpha^2/2} = \frac{\frac{g(¾\alpha)-g(¼\alpha)}{\alpha/2} - \frac{g(-¼\alpha)-g(-¾\alpha)}{\alpha/2}}{\alpha} \underset{\alpha\to 0}{\longrightarrow} \frac{g'(½\alpha)-g'(-½\alpha)}{\alpha} \underset{\alpha\to 0}{\longrightarrow} g''(0).$$

Since this is the expected sifting behavior of $\delta''(x)$, we conclude that, in the limit when $\alpha\to 0$, the function $f''(x)$ approaches $\delta''(x)$.